EULERIAN NUMBERS AND THE UNIT CUBE

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1. INTRODUCTION

There is an excellent expository paper [3] on Eulerian numbers and polynomials, and we begin with a quotation from it: "Following Euler [5] we may put

$$\frac{1-\lambda}{e^x-\lambda} = \sum_{n=0}^{\infty} H_n \frac{x^n}{n!} \qquad (\lambda \neq 1), \tag{1.1}$$

where $H_n = H_n(\lambda)$ is a rational function of λ; indeed

$$R_n = R_n(\lambda) = (\lambda - 1)^n H_n(\lambda) \tag{1.2}$$

is a polynomial in λ of degree $n - 1$ with integral coefficients. If we put

$$R_n = \sum_{k=1}^{n} A_{nk}\lambda^{k-1} \qquad (n \geqslant 1), \tag{1.3}$$

then the first few values of A_{nk} are given by the following table, where n denotes the row and k the column;

1						
1	1					
1	4	1				(1.4)
1	11	11	1			
1	26	66	26	1		
1	57	302	302	57	1	

Alternatively, Worpitzky showed that the A_{nk} may be defined by means of

$$x_n = \sum_{k=1}^{n} A_{nk}\binom{x+k-1}{n}.\text{"} \tag{1.5}$$

The numbers A_{nk} occur in connection with Bernoulli numbers and polynomials [11], and splines [10], and as the number of permutations of $(1, 2, \ldots, n)$ with k *rises*. [A permutation $(a_1, \ldots, a_n)$ has a rise at a_i if $a_i < a_{i+1}$; by convention, there is a rise to the left of a_1.] The A_{nk} satisfy a recursion and are symmetric:

$$A_{n+1,k} = kA_{n,k} + (n-k+1)A_{n,k-1} \tag{1.6}$$

and

$$A_{n,k} = A_{n,n-k+1} \qquad (1 \leqslant k \leqslant n).$$

From (1.6), it follows that

$$\sum_{k=1}^{n} A_k = n! \qquad (n \geqslant 1).$$

We now consider the unit cube $Q_n : 0 \leqslant x_i \leqslant 1$ $(1 \leqslant i \leqslant n)$, with the usual measure. It is evident from elementary calculations and from observation of (1.4) that, for $n = 2, 3,$ or 4 and $1 \leqslant k \leqslant n$, the volume V_{nk} of the section

$$k - 1 \leqslant \sum_{i=1}^{n} x_i \leqslant k$$

of the unit cube is given by $V_{nk} = A_{nk}/n!$. This observation led Hillman (in a private communication with this author) to conjecture that, generally,

$$V_{nk} = A_{nk}/n!$$

He was right.

2. APPLICATIONS

In the notation of Section 1, we have

THEOREM 1: For $1 \leqslant k \leqslant n$, we have $V_{nk} = A_{nk}/n!$ (2.1)

The proof is not difficult, but we defer that to the last. What is nice about this is that the unit cube is the natural probability space for a sum of n independent random variables X_i $(1 \leqslant i \leqslant n)$ identically and uniformly distributed on $[0, 1]$. Thus, we may reinterpret (2.1) to read:

$$\text{For } 1 \leqslant k \leqslant n, \ \text{Prob}\left(k - 1 \leqslant \sum_{1}^{n} X_i \leqslant k\right) = A_{nk}/n! \tag{2.2}$$

Through this interpretation, the central limit theorem and related results can be brought to bear on the asymptotic behavior of the Eulerian numbers.

For instance, the variance of each X_i is

$$\int_0^1 (x - 1/2)^2 dx = 1/12.$$

Thus the variance of $\sum_{1}^{n} X_i$ is $n/12$. Now, by the central limit theorem, if x is fixed and

$$\omega_n = (n/12)^{1/2} x + \frac{1}{2}n,$$

then

$$\lim_{n\to\infty} \text{Prob}\left(\sum_1^n X_i \leqslant \omega_n\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt. \tag{2.3}$$

Since the probability density function $f_n(t)$ of $\sum_1^n X_i$ tends to zero uniformly in t as $n \to \infty$, we can replace ω_n with $[\omega_n]$ in (2.3). Then, from (2.2), we have

$$\lim_{n\to\infty} \sum_{k=1}^{[\omega_n]} A_{nk}/n! = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt. \tag{2.4}$$

This is equivalent to Theorem 1 of [4]. It may be that this approach permits a simpler proof or an improvement in the error term in the other theorem of [4], which states that

$$(1/n!)A_{n,[\omega_n]} = (6/n\pi)^{1/2}\exp\left(-\frac{1}{2}x^2\right) + O(n^{-3/4}). \tag{2.5}$$

From a geometric point of view, one important property of the cube is that it is convex. The Brunn-Minkowski theorem states that the area $A(t)$ of the intersection of a hyperplane $H(t)$ with equation

$$\sum_1^n c_i x_i = t$$

with a convex body Q in real n-space has a concave nth root on the interval where it is positive. Thus, if $H_n(t)$ has equation

$$\sum_1^n x_i = t$$

and $A_n(t)$ is the area of $H_n(t) \cap Q_n$ (where Q_n is still the unit cube $0 \leqslant x_i \leqslant 1$, $1 \leqslant i \leqslant n$), then $(A_n(t))^{1/n}$ is concave on $(0, n)$. Consequently,

$$\log A_n(t) \text{ is concave on } (0, n). \tag{2.6}$$

There is a simple relation between $A_n(t)$ and the probability density function $f_n(t)$ of $\sum_1^n X_i$:

$$A_n(t) = \sqrt{n} f_n(t).$$

(See, e.g., [6].)

Now let V_{nk} be the volume of Q_n between $H(k - 1)$ and $H(k)$. Then,

$$V_{nk} = n^{-1/2}\int_{k-1}^{k} A_n(t)dt = \int_{k-1}^{k} f_n(t)dt. \tag{2.7}$$

There is a considerable literature on logarithmic concavity. A function $g(t)$ is called *log-concave* if $g(t) \geqslant 0$ on $\mathbf{R}$ and is positive on just one interval, and if $\log g(t)$ is concave on that interval. A very special case of a theorem due to Prekopa says that if $f(t)$ is log-concave, then

$$F(x) = \int_{x-c}^{x} f(t)dt$$

is also log-concave [2, 8, 9]. In particular,

$$V(x) = n^{-1/2}\int_{x-c}^{x} A(t)dt$$

is log-concave, and in most particular,

$$V_{n,\,k-1}V_{n,\,k+1} \leqslant V_{n,\,k}^2, \tag{2.8}$$

or what is the same thing,

$$A_{n,\,k-1}A_{n,\,k+1} \leqslant A_{n,\,k}^2. \tag{2.9}$$

This is due to Kurtz, who proved strict inequality in (2.9) when $1 \leqslant k \leqslant n$.

3. PROOF OF THEOREM 1

The probability density functions $f_n(t)$ for $\sum_1^n X_i$ can be generated recursively starting with

$$f_1(t) = \begin{cases} 1 & \text{if } 0 \leqslant t \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

and using

$$f_{n+1}(t) = f_n(t) * f_1(t) = \int_0^t f_n(u)f_1(t-u)du = \int_{t-1}^{t} f_n(u)du. \tag{3.1}$$

Thus,

$$V_{nk} = \int_{k-1}^{k} f_n(t)dt = f_{n+1}(k). \tag{3.2}$$

It follows from (1.5) (but not trivially) that

$$A_{nk} = \sum_{j=0}^{k-1} (-1)^j \binom{n+1}{j}(k-j)^n. \tag{3.3}$$

This is (2.15) of [3] and is due to Euler. Thus, we can prove Theorem 1 by showing that

$$f_{n+1}(k) = \frac{1}{n!}\sum_{j=0}^{k-1} (-1)^j \binom{n+1}{j}(k-j)^n. \tag{3.4}$$

Now, $f_{n+1}(t)$ is the convolution of $n + 1$ copies of $f_1(t)$, so its Laplace transform is

$$F(s) = \left(\frac{1}{s}(1 - e^{-s})\right)^{n+1}. \tag{3.5}$$

(See, e.g., [1].) Expanding (3.5) by the binomial theorem gives

$$F(s) = (1/s)^{n+1}\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}e^{-sj},$$

and the inverse Laplace transform of the sum of these $n+2$ terms computes to

$$f_{n+1}(t) = \sum_{j=0}^{n+1}\frac{1}{n!}(-1)^j\binom{n+1}{j}(t-j)_+^n, \tag{3.6}$$

where $(t-j)_+$ is 0 for $t < j$ and $t - j$ for $t \geqslant j$. With $t = k$, (3.6) reduces to (3.4). □

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