SOME PROPERTIES OF DIVISIBILITY OF HIGHER-ORDERED LINEAR RECURSIVE SEQUENCES

GERÖCS LÁSZLÓ
Balzac U. 35, Budapest, 1136, V.3. Hungary
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In this paper we consider the Fibonacci sequence defined by

$$F_0 = 0,\ F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2},\ n \geqslant 2,$$

the k-ordered Fibonacci sequence $\{G_n^{(k)}\}$, and the generalized k-ordered linear recursive sequence $\{R_n^{(k)}\}$, both of which will be defined.

First a new relation on the Fibonacci sequence will be proved and a well-known relation on the Fibonacci sequence will be generalized for the k-ordered Fibonacci sequence. Then an infinite set of positive integers will be found such that no integer in this set is a divisor of any term in the sequence $\{R_n^{(k)}\}$. Finally, a result of Lieuwens [1] will be generalized for k-ordered linear recursive sequences.

DEFINITION 1: For every $k > 1$, the k-ordered Fibonacci sequence $\{G_n^{(k)}\}$ is defined by $G_0^{(k)} = G_1^{(k)} = \cdots = G_{k-1}^{(k)} = 1$, and

$$G_n^{(k)} = \sum_{i=1}^{k} G_{n-i}^{(k)},\ n \geqslant k.$$

(When $k = 2$, this sequence is essentially the Fibonacci sequence.)

DEFINITION 2: For every $k > 1$, the generalized k-ordered linear recursive sequence $\{R_n^{(k)}\}$ is defined by $R_0^{(k)} = R_1^{(k)} = \cdots = R_{k-1}^{(k)} = 1$, and

$$R_n^{(k)} = \sum_{i=1}^{k} a_i R_{n-i}^{(k)},\ n \geqslant k,$$

where the a_i are integers not all equal to 0.

DEFINITION 3: If $m \neq 0$ is an integer, then for every $k > 1$, the length of the period modulo m of $\{R_n^{(k)}\}$ is the least natural number $p(m)$ such that there exists an index n_0, and for $n > n_0$,

$$R_{n+p}^{(k)} \equiv R_n^{(k)} \pmod{m}.$$

A sequence is called absolutely periodic modulo m if $n_0 = 0$.

REMARK: Every sequence $\{R_n^{(k)}\}$ is clearly periodic.

DEFINITION 4: The occurrence order of the natural number $m > 1$ in the sequence $\{R_n^{(k)}\}$ is the number $r(m)$, for which $m | R_r^{(k)}$, but $m \nmid R_n^{(k)}$ if $0 < n < r$.

EXAMPLE 1: Let the $a_i = 1$ and $k = 3$. Then we have the sequence

$$\{R_n^{(3)}\} \equiv 1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, \ldots .$$

If $m = 5$, this sequence reduced modulo 5 becomes

1, 1, 1, 3, 0, 4, 2, 1, 2, 0, 3, 0, 3, 1, 4, 3, 3, 0, 1, 4,
0, 0, 4, 4, 3, 1, 3, 2, 1, 1, 4, 1, 1, 1, 3, ...,

and we have

$$p(5) = 31,\ n_0 = 0,\ r(5) = 4.$$

THEOREM 1: *If $\{R_n\}$ is the sequence defined by*

$$R_0 = 1,\ R_n = \sum_{j=1}^{n} jR_{n-j},\ n > 0,$$

then for $n \geqslant 2$,

(a) $R_n = F_{2n}$;

(b) $\sum_{j=0}^{n} R_j = F_{2n+1}$.

PROOF: (a) For $n = 2$, 3, and 4, the theorem is easily established. Using finite induction, and assuming that for $i > 4$,

$$R_i = F_{2i},$$

then

$$\begin{aligned} F_{2(i+1)} &= F_{2i+2} = F_{2i+1} + F_{2i} = F_{2i} + F_{2i-1} + F_{2i} \\ &= 2F_{2i} + F_{2i} - F_{2i-2} = 3F_{2i} - F_{2(i-1)} = 3R_i - R_{i-1} \\ &= 3\sum_{j=1}^{i} jR_{i-j} - \sum_{j=1}^{i-1} jR_{i-j-1} = \sum_{j=1}^{i} (2j+1)R_{i-j} \\ &= \sum_{j=1}^{i} jR_{i-j} + \sum_{j=2}^{i+1} jR_{i+1-j} = R_i + \sum_{j=2}^{i+1} jR_{i+1-j} \end{aligned}$$

$$= \sum_{j=1}^{i+1} jR_{i+1-j} = R_{i+1},$$

as required.

(b) Applying (a) above, we have

$$F_{2n+1} = F_{2(n+1)} - F_{2n} = R_{n+1} - R_n$$
$$= \sum_{j=1}^{n+1} jR_{n+1-j} - \sum_{j=1}^{n} jR_{n-j} = \sum_{j=0}^{n} R_j.$$

A well-known identity for Fibonacci numbers is

$$F_n = \sum_{i=2}^{n} F_{n-i} + 1,\ n \geqslant 2. \tag{1}$$

An alternate form of (1), which we obtain by renaming $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, and generalize as Theorem 2, is

$$F_n = \sum_{i=2}^{n-2} F_{n-i} + 3,\ n \geqslant 4. \tag{2}$$

<u>*THEOREM 2:*</u> *If $G_n^{(k)}$ is as in Definition* 1, *then for all $n \geqslant 2k$,*

$$G_n^{(k)} = \sum_{i=1}^{k-2} iG_{n-i-1}^{(k)} + (k-1)\sum_{i=k}^{n-k} G_{n-i}^{(k)} + \frac{k(k+1)}{2}. \tag{3}$$

Note that $G_n^{(2)} = F_n$ as defined in (2) *and hence* (2) *is a special case of* (3).

<u>*PROOF:*</u> Let $k \geqslant 2$ be fixed. If $n = 2k$, then using the definition of $G_{2k}^{(k)}$ twice and performing the indicated sums, we have

$$G_{2k}^{(k)} = \sum_{i=1}^{k} G_{2k-i}^{(k)} = \sum_{i=1}^{k} \sum_{j=1}^{k} G_{2k-i-j}^{(k)}$$
$$= G_{2k-2}^{(k)} + 2G_{2k-3}^{(k)} + \cdots + (k-2)G_{k+1}^{(k)} + (k-1)G_k^{(k)} + \frac{k(k+1)}{2}$$
$$= \sum_{i=1}^{k-2} iG_{2k-i-1}^{(k)} + (k-1)G_k^{(k)} + \frac{k(k+1)}{2}.$$

(Recall that $G_0^{(k)} = G_1^{(k)} = \cdots = G_{k-1}^{(k)} = 1$.)

Now suppose that (3) is true for $m > 2k$. Then

$$G_{m+1}^{(k)} = \sum_{i=1}^{k} G_{m-i+1}^{(k)} = \sum_{i=0}^{k-1} G_{m-i}^{(k)} = G_m^{(k)} + \sum_{i=1}^{k-1} G_{m-i}^{(k)}$$
$$= \sum_{i=1}^{k-2} iG_{m-i-1}^{(k)} + (k-1)\sum_{i=k}^{m-k} G_{m-i}^{(k)} + \frac{k(k+1)}{2} + \sum_{i=1}^{k-1} G_{m-i}^{(k)}$$

$$= \left[\sum_{i=1}^{k-3} iG_{m-i-1}^{(k)} + \sum_{i=1}^{k-2} G_{m-i}^{(k)}\right] + \left[(k-2)G_{m-(k-1)}^{(k)} + G_{m-(k-1)}^{(k)} + (k-1)\sum_{i=k}^{m-k} G_{m-i}^{(k)}\right] + \frac{k(k+1)}{2}$$

$$= \sum_{i=1}^{k-2} iG_{(m+1)-i-1}^{(k)} + (k-1)\sum_{i=k}^{m+1-k} G_{(m+1)-i}^{(k)} + \frac{k(k+1)}{2},$$

which proves that (3) is true for $n = m + 1$ and hence for all n.

We now turn to the question of divisibiltiy of the terms of the sequence $\{R_n^{(k)}\}$ by the natural number m and state the following theorem.

<u>*THEOREM 3:*</u> *If $\{R_n^{(k)}\}$ is as in Definition* 2, *and if m if a natural number such that*

$$\left(\sum_{i=1}^{k} a_i\right) - 1 \neq 0$$

and

$$\text{g.c.d.}\left(m, \left(\sum_{i=1}^{k} a_i - 1\right)\right) = d > 1,$$

then $m \nmid R_n^{(k)}$ for any n. That is, $r(m)$ does not exist.

<u>*PROOF:*</u> Let

$$M = \left(\sum_{i=1}^{k} a_i\right) - 1.$$

If g.c.d. $(m, M) = d > 1$, we show that for every n,

$$R_n^{(k)} \equiv 1 \pmod{M}.$$

If $n < k$, then $R_n^{(k)} = 1$ and $M \nmid R_n^{(k)}$, since $M > 1$.

Now, if we assume that the theorem is true for any k successive terms of the sequence, we have

$$R_n^{(k)} = j_0 M + 1$$

$$R_{n+1}^{(k)} = j_1 M + 1$$

$$\cdots\cdots\cdots$$

$$R_{n+k-1}^{(k)} = j_{k-1} M + 1.$$

Multiplying each of these equations successively by $a_k, a_{k-1}, \ldots, a_1$, we obtain

$$\begin{aligned} a_k R_n^{(k)} &= a_k j_0 M + a_k \\ a_{k-1} R_{n+1}^{(k)} &= a_{k-1} j_1 M + a_{k-1} \\ &\cdots\cdots\cdots\cdots \\ a_1 R_{n+k-1}^{(k)} &= a_1 j_{k-1} M + a_1, \end{aligned}$$

and then adding, we have

$$\begin{aligned} R_{n+k}^{(k)} = \sum_{i=1}^{k} a_i R_{n+k-i}^{(k)} &= M \sum_{i=1}^{k} a_i j_{k-i} + \left(\sum_{i=1}^{k} a_i\right) - 1 + 1 \\ &= M\left(\sum_{i=1}^{k} a_i j_{k-i} + 1\right) + 1, \end{aligned}$$

which establishes that $R_{n+k}^{(k)} \equiv 1 \pmod{M}$.

Now we assume that for some s,

$$m | R_s^{(k)}.$$

Then $d | R_s^{(k)}$ and $d | M$ and hence there exist integers j, r_1, and r_2 such that

$$R_s^{(k)} = r_1 d = jM + 1 = r_2 d + 1,$$

which implies $d | 1$, a contradiction, and the proof is complete.

If g.c.d. $(m, M) = 1$, then it is not known whether, in general, there exists n such that $m | R_n^{(k)}$.

Finally, we examine $p(m)$, the length of the period of $\{R_n^{(k)}\}$ modulo m.

Waddill[2] has shown that in the special case where $R_0 = 0$, $R_1 = R_2 = 1$, $k = 3$, $a_1 = a_2 = a_3 = 1$, and $m = q_1^{\alpha_1}, q_2^{\alpha_2}, \ldots, q_r^{\alpha_r}$, q_i prime, then

$$p(m) = \text{l.c.m. } [p(q_1^{\alpha_1}), p(q_2^{\alpha_2}), \ldots, p(q_r^{\alpha_r})]. \tag{4}$$

Lieuwens [1] has shown that (4) holds for an arbitrary 2-ordered sequence. We show that (4) is true for every k-ordered sequence.

__THEOREM 4:__ *Let $\{R_n^{(k)}\}$ be as in Definition 2 and let $m > 1$ be an arbitrary integer, where*

$$m = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_r^{\alpha_r},\ q_i \text{ prime},$$

then

$$p(m) = \text{l.c.m. } [p(q_1^{\alpha_1}), p(q_2^{\alpha_2}), \ldots, p(q_r^{\alpha_r})].$$

PROOF: For every integer $q_i^{\alpha_i}$, there exists an index n_{0_i} such that for $n > n_{0_i}$,

$$R^{(k)}_{n+jp(q_i^{\alpha_i})} \equiv R^{(k)}_n \pmod{q_i^{\alpha_i}},\ j = 0,\ 1,\ 2,\ \ldots\ .$$

Let $n^* = \max(n_{0_1}, n_{0_2}, \ldots, n_{0_r})$. Then for every integer $t > 0$, $j \geqslant 0$,

$$R^{(k)}_{n^*+jp(q_i^{\alpha_i})+t} \equiv R^{(k)}_{n^*+t} \pmod{q_i^{\alpha_i}}$$

for all i. Hence, for $i = 1, 2$, say,

$$R^{(k)}_{n^*+jp(q_1^{\alpha_1})+t} \equiv R^{(k)}_{n^*+t} \pmod{q_1^{\alpha_1}}$$

$$R^{(k)}_{n^*+jp(q_2^{\alpha_2})+t} \equiv R^{(k)}_{n^*+t} \pmod{q_2^{\alpha_2}},$$

Since g.c.d. $(q_1, q_2) = 1$, then the smallest integer, p, such that

$$R^{(k)}_{n^*+p+t} \equiv R^{(k)}_{n^*+t} \pmod{q_1^{\alpha_1} q_2^{\alpha_2}}$$

occurs when

$$p = \text{l.c.m.}\ [p(q_1^{\alpha_1}),\ p(q_2^{\alpha_2})],$$

since p must be a multiple of both $p(q_1^{\alpha_1})$ and $p(q_2^{\alpha_2})$. The general case follows similarly.

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