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ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-349 *Proposed by Paul S. Bruckman, Carmichael, CA*

Define S_n as follows:

$$S_n \equiv \sum_{k=1}^{n-1} \csc^2 \pi k/n, \; n = 2, 3, \ldots .$$

Prove $S_n = \dfrac{n^2 - 1}{3}$.

H-350 *Proposed by M. Wachtel, Zürich, Switzerland*

There exist an infinite number of sequences, each of which has an infinite number of solutions of the form:

$$A \cdot x_1^2 + 1 = 5 \cdot y_1^2 \qquad \underline{A} = 5 \cdot (a^2 + a) + 1 \qquad \underline{a} = 0, 1, 2, 3, \ldots$$

$$A \cdot x_2^2 + 1 = 5 \cdot y_2^2 \quad \underline{x_1} = 2; \; \underline{x_2} = 40(2a + 1)^2 - 2$$

$$A \cdot x_3^2 + 1 = 5 \cdot y_3^2 \quad \underline{y_1} = 2a + 1; \; \underline{y_2} = (2a + 1) \cdot (16A + 1)$$

$$\ldots \qquad\qquad \ldots$$

$$A \cdot x_n^2 + 1 = 5 \cdot y_n^2$$

Find a recurrence formula for x_3/y_3, x_4/y_4, $\ldots$, x_n/y_n. (y_n is dependent on x_n.)

Examples

$\underline{a = 0}$ $\quad 1 \cdot \left(\dfrac{L_3}{2}\right)^2 + 1 = 5 \cdot \left(\dfrac{F_3}{2}\right)^2 \qquad$ $\underline{a = 1}$ $\quad 11 \cdot 2^2 + 1 = 5 \cdot 3^2$

$\underline{a = 0}$ $1 \cdot \left(\frac{L_9}{2}\right)^2 + 1 = 5 \cdot \left(\frac{F_9}{2}\right)^2$

$1 \cdot \left(\frac{L_{15}}{2}\right)^2 + 1 = 5 \cdot \left(\frac{F_{15}}{2}\right)^2$

$1 \cdot \ldots \quad + 1 = 5 \cdot \ldots$

$\underline{a = 1}$ $11 \cdot 358^2 + 1 = 5 \cdot 531^2$

$11 \cdot 63722^2 + 1 = 5 \cdot 94515^2$

$11 \cdot \ldots \quad + 1 = 5 \cdot \ldots$

$\underline{a = 5}$ $151 \cdot 2^2 + 1 = 5 \cdot 11^2$

$151 \cdot 4{,}838^2 + 1 = 5 \cdot 26{,}587^2$

$151 \cdot 11{,}698{,}282^2 + 1 = 5 \cdot 64{,}287{,}355^2$

$151 \quad \ldots \quad + 1 = 5 \quad \ldots$

H-351 *Proposed by Verner E. Hoggatt, Jr. (deceased)*

Solve the following system of equations:

$$\begin{aligned} U_1 &= 1 \\ V_1 &= 1 \\ U_2 &= U_1 + V_1 + F_2 = 3 \\ V_2 &= U_2 + V_1 = 4 \\ &\vdots \\ U_{n+1} &= U_n + V_n + F_{n+1} \qquad (n \geq 1) \\ V_{n+1} &= U_{n+1} + V_n \qquad (n \geq 1) \end{aligned}$$

SOLUTIONS

Eventually

H-332 *Proposed by David Zeitlin, Minneapolis, MN (Vol. 19, No. 4, October 1981)*

Let $\alpha = (1 + \sqrt{5})/2$. Let $[x]$ denote the greatest integer function. Show that after k iterations $(k \geq 1)$, we obtain the identity

$$[\alpha^{4p+2}[\alpha^{4p+2}[\alpha^{4p+2}[\ldots]]]] = F_{(2p+1)(2k+1)}/F_{2p+1} \qquad (p = 0, 1, \ldots).$$

Remarks: The special case $p = 0$ appears as line 1 in Theorem 2, p. 309, in the paper by Hoggatt & Bicknell-Johnson, this Quarterly, Vol. 17, No. 4, pp. 306-318. For $k = 2$, the above identity gives

$$[\alpha^{4p+2}[\alpha^{4p+2}]] = F_{5(2p+1)}/F_{2p+1} = L_{4(2p+1)} - L_{2(2p+1)} + 1.$$

Solution by Paul S. Bruckman, Carmichael, CA

We may proceed by induction on k. For brevity, let Φ_k denote

$$\underbrace{[\alpha^{4p+2}[\alpha^{4p+2}[\alpha^{4p+2}[\ldots]]]]}_{k \text{ pairs of brackets}}, \text{ considering } p \text{ fixed;}$$

we seek to prove that

$$\Phi_k = \frac{F_{(2p+1)(2k+1)}}{F_{2p+1}}, \quad k = 1, 2, 3, \ldots . \tag{1}$$

Let S denote the set of natural numbers k for which (1) holds. Note that

$$\Phi_1 = [\alpha^{4p+2}] = [L_{4p+2} - \beta^{4p+2}] = L_{4p+2} - 1,$$

since $0 < \beta^{4p+2} < 1$. Thus, $1 \in S$.

Suppose $k \in S$. Then

$$\Phi_{k+1} = \left[\frac{\alpha^{4p+2}F_{(2p+1)(2k+1)}}{F_{2p+1}}\right],$$

under the inductive hypothesis.

Now if m and n are odd, with $n \geqslant 3$, then

$$\alpha^{2m}F_{mn}/F_m = \alpha^{2m}(\alpha^{mn} - \beta^{mn})/F_m\sqrt{5} = \frac{\alpha^{m(n+2)} - \beta^{m(n-2)}}{\sqrt{5}F_m}$$

$$= \frac{\alpha^{m(n+2)} - \beta^{m(n+2)} - \beta^{mn}(\alpha^{2m} - \beta^{2m})}{\sqrt{5}F_m} = \frac{F_{m(n+2)}}{F_m} - \beta^{mn}L_m.$$

Since $-1 < \beta^{mn} < 0$, $\beta^{mn}L_m < 0$. Also,

$$-\beta^{mn}L_m = \alpha^{-mn}(\alpha^m - \alpha^{-m}) = \alpha^{-m(n-1)} - \alpha^{-m(n+1)} < \alpha^{-m(n-1)} \leqslant \alpha^{-2} < 1.$$

Therefore, $0 < -\beta^{mn}L_m < 1$, which implies

$$\left[\frac{\alpha^{2m}F_{mn}}{F_m}\right] = \frac{F_{m(n+2)}}{F_m}. \tag{2}$$

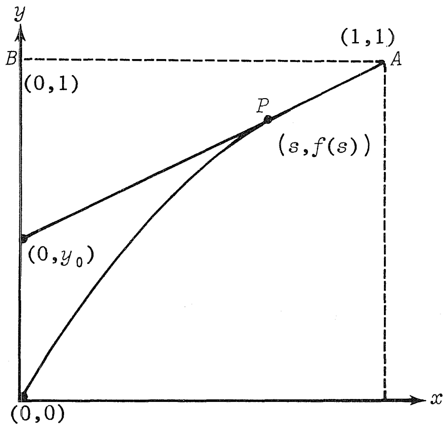
Setting $m = 2p + 1$, $n = 2k + 1$ in (2), this is equivalent to the assertion of (1) for $k + 1$. Since $k \in S \to (k + 1) \in S$, the proof by induction follows at once.

Also solved by the proposer.

Nab That Pig

H-333 *Proposed by Paul S. Bruckman, Carmichael, CA*
(Vol. 19, No. 5, December 1981)

The following problem was suggested by Problem 307 of *536 Puzzles & Curious Problems*, by Henry Ernest Dudeney, edited by Martin Gardner (New York: Charles Scribner's Sons, 1967).



Leonardo and the pig he wishes to catch are at points A and B, respectively, one unit apart (which we may consider some convenient distance, e.g., 100 yards). The pig runs straight for the gateway at the origin, at uniform speed. Leonardo, on the other hand, goes directly toward the pig at all times, also at a uniform speed, thus taking a curved course. What must be the ratio r of Leonardo's speed to the pig's, so that Leonardo may catch the pig just as they both reach the gate?

Solution by the proposer

Let the curve along which Leonardo runs be represented by the equation

$$y = f(x). \tag{1}$$

We note that f must be continuously differentiable in $(0, 1)$ and that the following additional conditions are to be satisfied:

$$f(1) = 1; \tag{2}$$

$$f'(1) = 0; \tag{3}$$

$$f(0) = 0. \tag{4}$$

The tangent line at any point $P \equiv (s, f(s))$ of the curve has the equation: $y - f(s) = f'(s)(x - s)$, with y-intercept $y_0 = f(s) - sf'(s)$. Thus, the distance the pig has traveled when Leonardo is at point P is equal to $1 - y_0 = 1 - f(s) + sf'(s)$. On the other hand, the distance Leonardo has traveled at that point is equal to

$$\int_s^1 \sqrt{1 + (f'(t))^2}\, dt,$$

as is well known from the calculus.

With a change of notation, this implies the relationship:

$$\int_x^1 \sqrt{1 + (f'(t))^2}\, dt = r(1 - f(x) + xf'(x)), \tag{5}$$

which is to be satisfied, along with (2), (3), and (4).

We may differentiate each side of (5) with respect to x (assuming this to be legitimate), thereby obtaining

$$-\sqrt{1 + (f'(x))^2} = rxf''(x),$$

or equivalently:

$$\frac{f''(x)}{\sqrt{1 + (f'(x))^2}} = -\frac{1}{rx}. \tag{6}$$

Integrating each side of (6) and using (3), we find that

$$\log\left\{f'(x) + \sqrt{1 + (f'(x))^2}\right\} = -\frac{1}{r}\log x,$$

or

$$\sqrt{1 + (f'(x))^2} + f'(x) = x^{-1/r}. \tag{7}$$

Solving for $f'(x)$ in (7) (by transposing and squaring), we obtain:

$$f'(x) = \tfrac{1}{2}(x^{-1/r} - x^{1/r}). \tag{8}$$

Now integrating (8) and using (2), this yields:

$$f(x) = \frac{1}{2}\left\{\frac{x^{1-1/r}}{1-1/r} - \frac{x^{1+1/r}}{1+1/r}\right\} + C$$

$$= \frac{r}{2(r^2 - 1)}\left\{(r + 1)x^{1-1/r} - (r - 1)x^{1+1/r}\right\} + C,$$

where $f(1) = 1 = \dfrac{r}{r^2 - 1} + C$; hence, $C = (r^2 - r - 1)/(r^2 - 1)$, and

$$f(x) = \frac{2(r^2 - r - 1) + r(r + 1)x^{1-1/r} - r(r - 1)x^{1+1/r}}{2(r^2 - 1)}. \tag{10}$$

In order for Leonardo to catch his pig, it is clearly necessary that $r > 1$. We need to determine the particular value(s) of r satisfying (4), with $r > 1$. Setting $x = 0$ in (10), and assuming $f(0) = 0$ and $r > 1$, we obtain the equation $r^2 - r - 1 = 0$, whose only admissible solution is

$$r = \alpha = \tfrac{1}{2}(1 + \sqrt{5}), \text{ the Golden Mean.} \tag{11}$$

If $r > \alpha$, Leonardo will catch the pig before reaching the gate, while if $r < \alpha$, the pig will escape.

NOTE: In the original problem Dudeney gives the value $r = 2$ and asks for $f(0)$, which turns out to be 1/3.

CHECK: Substituting the value $r = \alpha$ in (10), we obtain:

$$f(x) = \frac{\alpha^3 x^{1-1/\alpha} + \alpha\beta x^{1+1/\alpha}}{2\alpha}$$

or equivalently:

$$f(x) = \frac{\alpha^2 x^{\beta^2} + \beta x^{\alpha}}{2}, \text{ where } \beta = \tfrac{1}{2}(1 - \sqrt{5}). \tag{12}$$

The distance that the pig runs to the gate is, of course, 1. We should thus find that the length of the curve from (0, 0) to (1, 1) (call this distance d) is equal to α. Now

$$d = \int_0^1 \sqrt{1 + (f'(x))^2}\, dx.$$

Differentiating (12), we obtain:

$$f'(x) = \tfrac{1}{2}\left\{\alpha^2\beta^2 x^{\beta^2-1} + \alpha\beta x^{\alpha-1}\right\} = \tfrac{1}{2}(x^{\beta} - x^{-\beta});$$

$$1 + (f'(x))^2 = \left\{\tfrac{1}{2}(x^{\beta} + x^{-\beta})\right\}^2;$$

and

$$d = \tfrac{1}{2}\int_0^1 (x^{\beta} + x^{-\beta})dx = \tfrac{1}{2}\left(\frac{x^{1+\beta}}{1+\beta} + \frac{x^{1-\beta}}{1-\beta}\right)\Bigg|_0^1 = \tfrac{1}{2}(\alpha^2 x^{\beta^2} - \beta x^{\alpha})\Bigg|_0^1$$

$$= \tfrac{1}{2}(\alpha^2 - \beta) = \tfrac{1}{2}(\alpha + 1 - \beta) = \alpha,$$

as expected. The other conditions on f are readily verified for the function given by (12).

Also solved by B. Cheng.

Little Residue

H-334 *Proposed by Lawrence Somer, Washington, D.C.*
(Vol. 19, No. 5, December 1981)

Let the Fibonacci-like sequence $\{H_n\}_{n=0}^{\infty}$ be defined by the relation

$$H_{n+2} = aH_{n+1} + bH_n,$$

where a and b are integers, $(a, b) = 1$, and $H_0 = 0$, $H_1 = 1$. Show that if p is an odd prime such that $-b$ is a quadratic nonresidue of p, then

$$p \nmid H_{2n+1} \text{ for any } n \geqslant 0.$$

(This is a generalization of Problem B-224, which appeared in the December 1971 issue of this Quarterly.)

Solution by the proposer

I offer three solutions.

First Solution: It can be shown by induction or by the Binet formula that

$$H_{2n+1} = bH_n^2 + H_{n+1}^2.$$

Suppose that $p|H_{2n+1}$ and $(-b/p) = -1$. Since

$$(n,\ 2n+1) = (n+1,\ 2n+1) = 1,$$

$p \nmid H_n$ and $p \nmid H_{n+1}$. This follows because $\{H_n\}$ is periodic modulo p and because $H_0 = 0$. Thus,

$$bH_n^2 + H_{n+1}^2 \equiv 0 \pmod p$$

and

$$H_{n+1}^2 \equiv -bH_n^2 \pmod p.$$

Since neither H_n nor $H_{n+1} \equiv 0 \pmod p$ and since $(-b/p) = -1$, this is a contradiction.

Second Solution: It can be shown by the Binet formula or by induction that

$$H_n^2 - H_{n-1}H_{n+1} = (-b)^{n-1}.$$

Suppose $p|H_{2n+1}$ and $(-b/p) = -1$. Then it follows that

$$H_{2n+2}^2 - H_{2n+1}H_{2n+3} \equiv H_{2n+2}^2 \equiv (-b)^{2n+1} \pmod p.$$

Since $(-b/p) = -1$, this is a contradiction.

Third Solution: Let $\{J_n\}_{n=0}^{\infty}$ be defined by

$$J_{n+2} = aJ_{n+1} + bJ_n,$$

with $J_0 = 2$ and $J_1 = a$. It can be shown by the Binet formulas that

$$J_n^2 - (a^2 + 4b)H_n^2 = 4(-b)^n.$$

Suppose that $p|H_{2n+1}$ and $(-b/p) = -1$. Then

$$J_{2n+1}^2 - (a^2 + 4b)H_{n+1}^2 \equiv J_{2n+1}^2 \equiv 4(-b)^{2n+1}.$$

Since $(-b/p) = -1$, this is a contradiction.

Also solved by A. Shannon and P. Bruckman.

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