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INJECTIVITY OF EXTENDED GENERALIZED FIBONACCI SEQUENCES

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INTRODUCTION

Let N denote the set of positive integers, and Z the set of all integers. The function $F : Z \to Z$ with $F(1) = 1$, $F(2) = 1$, and $F(n) = F(n-2) + F(n-1)$ for every $n \in Z$, constitutes the extention to the left of the original Fibonacci sequence, where the domain is restricted to N. With the arguments written as subscripts, the following table gives the "middle" section of this extended two-sided sequence:

...	F_{-7}	F_{-6}	F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	...
	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13	

Similarly, one obtains the extended Lucas sequence as $L : Z \to Z$ with $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-2} + L_{n-1}$ for every $n \in Z$:

...	L_{-7}	L_{-6}	L_{-5}	L_{-4}	L_{-3}	L_{-2}	L_{-1}	L_0	L_1	L_2	L_3	L_4	L_5	L_6	L_7	...
	-29	18	-11	7	-4	3	-1	2	1	3	4	7	11	18	29	

In general, $H : Z \to Z$ with $H_1 = a \in Z$, $H_2 = b \in Z$, and $H_n = H_{n-2} + H_{n-1}$ for every $n \in Z$, constitutes the extended generalized Fibonacci sequence generated by the ordered pair of integers (a, b):

...	H_{-4}	H_{-3}	H_{-2}	H_{-1}	H_0	H_1	H_2	H_3	H_4	...
	$-8a + 5b$	$5a - 3b$	$-3a + 2b$	$2a - b$	$-a + b$	a	b	$a + b$	$a + 2b$	

The functions F and L as defined above are not injective; i.e., there are different arguments having the same values, or, in the terminology of sequences, some terms with different indices are equal, or, simpler still,

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some terms occur more than once. An extended generalized Fibonacci sequence generated by (a, b) will be called *injective* or *noninjective* according as the function H is injective or not.

The problem posed and solved in this paper is:

> *What are the necessary and sufficient conditions for* a *and* b *to generate an* injective *extended generalized Fibonacci sequence?*

For the sake of brevity, in this paper the word "sequence" stands for an "extended generalized Fibonacci sequence." In general such a sequence will be denoted by H with the values of H_1 and H_2 given, while F is short for H with $H_1 = H_2 = 1$ and L is short for H with $H_1 = 1$, $H_2 = 3$. (Since F commemorates Fibonacci and L commemorates Lucas, perhaps H might commemorate Hoggatt.)

In Section 1, it is proved that injective sequences do *exist*, which makes the research meaningful.

In Section 2, the problem is reduced to the investigation of a certain subset of the set of all sequences, a subset which *represents* all "candidates" for injectivity.

In Section 3, the *solution* is given.

Without further reference, some well-known properties will be used, e.g.:

$H_n = aF_{n-2} + bF_{n-1}$ for every $n \in Z$, where $a = H_1$ and $b = H_2$;

$L_n = F_{n-1} + F_{n+1}$ for every $n \in Z$;

$F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$ for every $n \in N \cup \{0\}$;

and finally, if two of any three consecutive terms of a sequence are known, then the whole sequence is known.

For these properties, see, e.g., [3; 5; 2].

1. EXISTENCE

The ordered pair (1, 1) generates F, but so does the ordered pair (1, 2), and any ordered pair of consecutive terms of F. The generated sequences are identical, the order-preserving shift of the indices is irrelevant. The pair (1, 3) generates L, and continuing along this line, one might consider the sequences generated by (1, 4), (1, 5), (1, 6), ... The following table shows that the first four sequences are noninjective, but from there on there seem to be candidates for injectivity, a conjecture strengthened by the use of a computer.

		H_{-7}	H_{-6}	H_{-5}	H_{-4}	H_{-3}	H_{-2}	H_{-1}	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	
	...	H_{-7}	H_{-6}	H_{-5}	H_{-4}	H_{-3}	H_{-2}	H_{-1}	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	...
F	...	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13	...
L	...	-29	18	-11	7	-4	3	-1	2	1	3	4	7	11	18	29	...
(1, 4)	...	-50	31	-19	12	-7	5	-2	3	1	4	5	9	14	23	37	...
(1, 5)	...	-71	44	-27	17	-10	7	-3	4	1	5	6	11	17	28	45	...
(1, 6)	...	-92	57	-35	22	-13	9	-4	5	1	6	7	13	20	33	53	...
(1, 7)	...	-113	70	-43	27	-16	11	-5	6	1	7	8	15	23	38	61	...
(1, 8)	...	-134	83	-51	32	-19	13	-6	7	1	8	9	17	26	43	69	...

Further inspection of the table suggests the following lemma.

Lemma 1

If $H_1 = 1$ and $H_2 = b \geqslant 3$, then $H_n = L_n + (b - 3)F_{n-1}$ for every $n \in Z$.

Proof: Every $n \in Z$ yields the identity

$$\begin{aligned} H_n &= F_{n-2} + bF_{n-1} \\ &= F_{n-2} + 3F_{n-1} + (b - 3)F_{n-1} \\ &= F_n + 2F_{n-1} + (b - 3)F_{n-1} \\ &= F_{n+1} + F_{n-1} + (b - 3)F_{n-1} \\ &= L_n + (b - 3)F_{n-1}. \end{aligned}$$

Another fact revealed by the table is the importance of the terms with even negative index. These are the terms that might be equal to terms with positive index.

Lemma 2

If $H_1 = 1$ and $H_2 = b \geqslant 3$, then $H_{-n} = H_n + (b - 3)F_n$ for every even $n \in Z$.

Proof: Let $n \in N$ be even. By Lemma 1, $H_{-n} = L_{-n} + (b - 3)F_{-n-1}$. Since n is even, $L_{-n} = L_n$, and $-n - 1$ is odd, so that $F_{-n-1} = F_{n+1}$. Hence

$$H_{-n} = L_n + (b - 3)F_{n+1}$$

or

$$H_{-n} = L_n + (b - 3)F_{n-1} + (b - 3)F_n .$$

Hence, by Lemma 1,

$$H_{-n} = H_n + (b - 3)F_n.$$

Since the sequence H generated by (1, 6) is the first candidate for being injective, a close inspection of this sequence is helpful. It is obvious that the sequence consists exhaustively of three one-sided sequences:

(i) the sequence H_n, $n \in N$, the strictly increasing sequence

$$1, 6, 7, 13, \ldots;$$

(ii) the sequence H_{-2n}, $n \in N \cup \{0\}$, the strictly increasing sequence

$$5, 9, 22, 57, \ldots;$$

(iii) the sequence $H_{-(2n-1)}$, $n \in N$, the strictly decreasing sequence

$$-4, -13, -35, -92, \ldots .$$

The only possibility for H to be noninjective is that the sequences (i) and (ii) have a common term.

Theorem 1

If $H_1 = 1$ and $H_2 = 6$, then H is injective.

Proof: Assume that H is noninjective. Then, by the introductory remarks above and by Lemma 2, there are $n \in N$ and $p \in N$, where $0 < n < p$ and n even, such that $H_{-n} = H_p$ with $H_p = H_n + 3F_n$. Since

$$H_p = F_{p-2} + 6F_{p-1} \quad \text{and} \quad H_n = F_{n-2} + 6F_{n-1},$$

one obtains

$$F_{n-2} + 6F_{n-1} + 3F_n = F_{p-2} + 6F_{p-1},$$

and hence

$$5F_{n-1} + 4F_n = F_p + 5F_{p-1},$$

which yields

$$4F_{n+1} + F_{n-1} = 4F_{p-1} + F_{p+1},$$

which gives

$$3F_{n+1} + L_n = 3F_{p-1} + L_p,$$

which finally results in

$$3(F_{n+1} - F_{p-1}) = L_p - L_n. \tag{1}$$

Since $0 < n < p$, one obtains $L_p > L_n$ and $L_p - L_n$ is positive. Therefore, $F_{n+1} - F_{p-1}$ is also positive, and hence $F_{n+1} > F_{p-1}$ and $n + 1 > p - 1$ or $p < n + 2$, which combined with $n < p$ yields $p = n + 1$. Rewriting the identity (1) with $n + 1$ for p, one obtains

$$3(F_{n+1} - F_n) = L_{n+1} - L_n,$$

or

$$3F_{n-1} = L_{n-1}. \tag{2}$$

Since $n \geqslant 2$ and $n = 2$ gives the contradiction $3F_1 = L_1$ or $3 = 1$, one obtains $n > 2$.

Next, it is proved by induction on n that $L_n < 3F_n$ for *every* $n > 2$.

Base step: $n = 3$ yields $L_3 = 4 < 6 = 3F_3$.

Induction step: Assume $L_k < 3F_k$ for every $k \in N$, $3 \leqslant k < m$, $m \in N$. Then $L_{m-2} + L_{m-1} < 3F_{m-2} + 3F_{m-1}$, by assumption; or $L_m < 3F_m$. Hence, by induction, $L_n < 3F_n$ for every $n \in N$, $n > 2$. Thus, for every even $n > 2$, certainly $L_{n-1} < 3F_{n-1}$, contrary to (2). Hence, by *reductio ad absurdum*, H is injective.

Corollary

If $H_1 = 1$ and $H_2 = b \geqslant 6$, then H is injective.

Proof: Assuming again that H is noninjective, one obtains the identity:

$$(b - 3)(F_{n+1} - F_{p-1}) = L_p - L_n, \tag{3}$$

which again yields $p = n + 1$, because $(b - 3) > 0$. Substituting $n + 1$ for p in (3), one obtains

$$(b - 3)F_{n-1} = L_{n-1}. \tag{4}$$

Again, $n = 2$ is contradictory, and for $n > 2$, the proof of Theorem 1 arrived at $L_{n-1} < 3F_{n-1}$, and therefore, since $b \geqslant 6$, certainly

$$L_{n-1} < (b - 3)F_{n-1},$$

contrary to (4).

2. REPRESENTATION

The search for the necessary and sufficient conditions for (a, b) to generate an injective sequence is simplified in two ways: (i) by elimination of classes of sequences which are obviously or can be proved to be noninjective; (ii) by representation of the remaining set of sequences by a proper subset so that the investigation may be restricted to that subset.

Trivially, all sequences generated by a pair of equal integers are noninjective. Moreover, if $a \neq b$, but either $a = 0$ or $b = 0$, then the sequences generated by (a, b) are noninjective; if $a = 0$, then $H_2 = H_3 = b$

and if $b = 0$, then $H_1 = a = H_3$. As a matter of fact, all sequences containing 0 can be discarded, since the successor and predecessor of that 0 are equal.

The sequence obtained from the sequence H by multiplication of all its terms by $c \in Z$ may be called a multiple of H and denoted by cH. If H is generated by (a, b), then cH is generated by (ca, cb). Clearly, cH is injective if and only if H is injective. If a and b are relatively prime, then the sequence generated by (a, b) represents all its multiples with respect to injectivity. This implies that the search can be restricted to generating pairs (a, b), where a and b are relatively prime. Moreover, the sequence $-H$, short for $(-1)H$, can be seen as the negative of H and clearly only one of H and $-H$ has to be considered.

As is well known (see [1], and also [4]), every sequence has two parts: a right-hand part where all the terms have the same sign (the *monotonic* portion) and a left-hand part where the signs of the terms alternate (the *alternating* portion). Let a sequence be called positive or negative according as the monotonic portion has positive or negative signs. Since any sequence can be generated by any successive pair of its terms, the search for injectivity can be restricted to pairs (a, b) where a and b have the same sign. Moreover, since a negative sequence is the negative of a positive sequence, a further restriction can be made to pairs (a, b) where a and b are both positive. In a positive sequence there is a *last* alternating pair; namely, the pair (H_{i-2}, H_{i-1}), $i \in Z$, where $H_{i-2} < 0$, $H_{i-1} > 0$, and $H_i > 0$ (in general, $H_i = 0$ is possible, as in F, but these sequences have already been discarded as noninjective). If the pair (H_{i-2}, H_{i-1}) is the last alternating pair of the sequence, the pair (H_i, H_{i+1}) may be called the *characteristic pair* of the sequence. It is the *unique* pair of successive terms of a positive sequence such that:

(i) $H_i > 0$ is the smallest term of the monotonic portion of the sequence;
(ii) H_i is the only term of the monotonic portion that is smaller than its predecessor;
(iii) H_i is the unique term of the monotonic portion that is smaller than half its successor.

As to (i), $H_i < H_{i-1}$ because $H_i - H_{i-1} = H_{i-2} < 0$, and $H_i < H_j$, for every $j > i$, because $H_j = H_i$ + one or more positive numbers. As to (ii), if there is another term in the monotonic portion smaller than its predecessor, say H_k, then $k > i$, since $k = i - 1$ does not qualify; but then $H_{k-2} = H_k - H_{k-1}$ would be negative and (H_{i-2}, H_{i-1}) would not be the last alternating pair. As to (iii), in general, for every $m \in Z$, $2H_m < H_{m+1}$ if and only if $H_m < H_{m-1}$. The argument is as follows: since

$$2H_n = 2aF_{n-2} + 2bF_{n-1} \quad \text{and} \quad H_{n+1} = aF_{n-1} + bF_n,$$

one obtains $2H_n < H_{n+1}$ if and only if

$$2aF_{n-2} + 2bF_{n-1} < aF_{n-1} + bF_n,$$

which holds if and only if

$$aF_{n-2} + bF_{n-1} < a(F_{n-1} - F_{n-2}) + b(F_n - F_{n-1}),$$

which is the same as

$$H_n < aF_{n-3} + bF_{n-2} \quad \text{or} \quad H_n < H_{n-1}.$$

It follows that H_i is the unique term of the monotonic portion such that $2H_i < H_{i+1}$, since H_i is the only term of the monotonic portion smaller than its predecessor.

Since every positive sequence has a characteristic pair, this pair can be seen to generate the sequence, and the investigation may be restricted further to pairs (a, b) where $2a < b$.

Summarizing, the investigation may be restricted to ordered pairs of integers (a, b), where $a \neq b$, both $a > 0$ and $b > 0$, a and b relatively prime and, finally, $2a < b$.

3. CONCLUSIONS

The following lemma is strongly suggested, of course, by the table in Section 1.

Lemma 3

Let $H_1 = a$, $H_2 = b$, and $0 < 2a < b$. Then $H_{-n} > 0$ for every even $n \in N$ and $H_{-n} < 0$ for every odd $n \in N$.

Proof: By induction on n.

Base step: $H_{-1} = 2a - b = -(b - 2a) < 0$;

$$H_{-2} = 2b - 3a = (b - 2a) + (b - a) > 0.$$

Induction step: Assume the lemma holds for all $k < m$, $k \in N$, $m \in N$, $m > 2$. If m is odd, then $m - 1$ is even and $m - 2$ odd; hence, by assumption, $H_{-(m-1)} > 0$ and $H_{-(m-2)} < 0$, or, $H_{-m+1} > 0$ and $H_{-m+2} < 0$. Since $H_{-m} = H_{-m+2} - H_{-m+1}$, it follows that $H_{-m} < 0$. Similarly, if m is even, $H_{-m+1} < 0$ and $H_{-m+2} > 0$, so that $H_{-m} = H_{-m+2} - H_{-m+1}$ yields $H_{-m} > 0$.

Conclusion: The lemma holds for every $n \in N$, by induction.

Corollary

Let H be as in Lemma 3. Then H is noninjective if and only if there exist $p \in N$ and even $n \in N$ such that $H_{-n} = H_p$.

Proof: Consider the set $\{H_m : m \in Z\}$ and its subsets $A = \{H_m : m > 0\}$, $B = \{H_m : m < 0,\ m \text{ odd}\}$, and $C = \{H_m : m \leqslant 0,\ m \text{ even}\}$. It can be readily shown, by considerations similar to those preceding Theorem 1, that within each of A, B, and C, one has $H_i = H_j$ if and only if $i = j$. Since A and B are clearly disjoint, and B and C also, for H to be noninjective, it is necessary that $A \cap C \neq \emptyset$. Since $H_0 = b - a$ is clearly not in A, it is then necessary that there exist an even $n \in N$ and $p \in N$ such that $H_{-n} = H_p$. Obviously, $H_{-n} = H_p$, $n \in N$, n even, $p \in N$, is sufficient for H to be noninjective.

Theorem 2

Let $H_1 = a$, $H_2 = b$, and $0 < 2a < b$. Moreover, let $H_{-n} = H_p$ for some even $n \in N$ and $p \in N$. Then $n - 2 < p < n + 2$.

Proof: If $H_{-n} = H_p$, then

$$aF_{-n-2} + bF_{-n-1} = aF_{p-2} + bF_{p-1},$$

or, since $n + 2$ is even and $n + 1$ is odd,

$$bF_{n+1} = a(F_{p-2} + F_{n+2}) + bF_{p-1}.$$

Further, $n > 0$ yields $F_{n+2} > 0$, and $p > 0$ yields $F_{p-2} \geqslant 0$, so that

$$a(F_{p-2} + F_{n+2}) > 0,$$

and hence

$$bF_{n+1} > bF_{p-1} \quad \text{or} \quad F_{n+1} > F_{p-1}.$$

Thus $n + 1 > p - 1$ or $p < n + 2$. On the other hand,

$$aF_{-n-2} + bF_{-n-1} = aF_{p-2} + bF_{p-1}$$

yields

$$b(F_{n+1} - F_{p-1}) = a(F_{p-2} + F_{n+2})$$

and, since $2a < b$, one obtains

$$a(F_{p-2} + F_{n+2}) > 2a(F_{n+1} - F_{p-1}).$$

Hence,

$$a(F_{p-2} + F_{p-1}) + aF_{n+2} + aF_{p-1} > 2aF_{n+1},$$

which yields

$$a(F_p + F_{p-1}) + aF_{n+2} > 2aF_{n+1}$$

or

$$aF_{p+1} + aF_{n+2} > 2aF_{n+1},$$

or

$$a(F_{n+2} - F_{n+1}) - aF_{n+1} > -aF_{p+1},$$

or

$$aF_n - aF_{n+1} > -aF_{p+1},$$

or

$$-aF_{n-1} > -aF_{p+1},$$

or

$$F_{n-1} < F_{p+1},$$

which in its turn yields $n - 1 < p + 1$ or $n - 2 < p$.

Remark: The next proof uses a little lemma of number theory which can be formulated as follows: let k and m be fixed positive integers generating the set of pairs $\{\{k, m+sk\} : s \in Z\}$; if one of these pairs is relatively prime, then each of them is. (The proof can be seen readily by consideration of the contrapositive: if one of the pairs has a common factor other than 1, then each pair has.) This lemma is applied four times in the proof of the next theorem, where a and b are fixed positive integers which are relatively prime:

1. If b is odd, then $2a$ and b are relatively prime also and hence, by the lemma, so are $2a$ and $b - 2a$.
2. If b is even, then $b - 2a$ is even and $2a$ and $b - 2a$ are not relatively prime, but a and $\frac{1}{2}b$ are, and hence, by the lemma, a and $\frac{1}{2}b - a$ are.
3. If a is odd, then a and $2b$ are relatively prime and hence, by the lemma, so are a and $2b - 3a$.
4. If a is even, then a and $2b - 3a$ have 2 as a common factor, but a and b are relatively prime and hence, by the lemma, so are $\frac{1}{2}a$ and $b - \frac{3}{2}a$.

Theorem 3

Let $H_1 = a$, $H_2 = b$, $0 < 2a < b$ and, moreover, let a and b be relatively prime. Then H is noninjective if and only if one of the following alternatives holds:

a. $a = 1$ and $b = 3$.

b. For some even $n > 0$, $2a = F_{n-1}$ and $b = F_{n+1}$, where b is odd.

c. For some even $n > 0$, $a = F_{n-1}$ and $\frac{1}{2}b = F_{n+1}$, where b is even (and hence a is odd).

d. For some even $n > 0$, $a = F_{n-1}$ and $2(b-a) = F_{n+1}$, where a is odd.

e. For some even $n > 0$, $\frac{1}{2}a = F_{n-1}$ and $b - a = F_{n+1}$, where a is even (and hence b is odd).

Proof: (i) If H is noninjective, then, by the corollary to Lemma 3, there exist $p \in N$ and even $n \in N$ such that $H_{-n} = H_p$. Then, by the previous theorem, $n - 2 < p < n + 2$. In the proof of the latter theorem, the identity

$$b(F_{n+1} - F_{p-1}) = a(F_{p-2} + F_{n+2}) \tag{5}$$

did appear, which will be used again here.

Case 1. $p = n$ yields for (5) the identity:

$$b(F_{n+1} - F_{n-1}) = a(F_{n-2} + F_{n+2}),$$

which transforms into

$$b(F_{n+1} - F_n + F_n - F_{n-1}) = a(F_n + F_{n+1} + F_{n-2}),$$

and hence

$$bF_n = a(F_{n-1} + F_{n+1} + 2F_{n-2}),$$

which yields

$$bF_n = a(2F_{n-1} + F_n + 2F_{n-2}),$$

and hence, finally,

$$bF_n = 3aF_n,$$

or, since $n > 0$ and hence $F_n \neq 0$,

$$b = 3a.$$

Since a and b are relatively prime, $a = 1$ and $b = 3$, which is alternative a.

Case 2. $p = n + 1$ yields for (5) the identity:

$$b(F_{n+1} - F_n) = a(F_{n-1} + F_{n+2}),$$

which gives

$$bF_{n-1} = a(F_{n-1} + F_n + F_{n+1}),$$

and hence

$$bF_{n-1} = a(2F_{n-1} + 2F_n),$$

which transforms into

$$(b - 2a)F_{n-1} = 2aF_n,$$

or, since $n \geqslant 2$ and hence $F_{n-1} \neq 0$,

$$\frac{F_n}{F_{n-1}} = \frac{b - 2a}{2a}.$$

Any two successive Fibonacci numbers are known to be relatively prime (cf, e.g. [3], p. 40). If b is odd, $2a$ and $b - 2a$ are relatively prime (remark 1), and hence, for some even $n > 0$,

$$2a = F_{n-1} \quad \text{and} \quad b - 2a = F_n,$$

or

$$2a = F_{n-1} \quad \text{and} \quad b = F_{n+1},$$

which is alternative b. If b is even, then

$$\frac{F_n}{F_{n-1}} = \frac{\frac{1}{2}b - a}{a}$$

and, since a and $\frac{1}{2}b - a$ are relatively prime (remark 2), for some even $n > 0$,

$$a = F_{n-1} \quad \text{and} \quad F_n = \tfrac{1}{2}b - a,$$

or

$$a = F_{n-1} \quad \text{and} \quad \tfrac{1}{2}b = F_{n+1},$$

which is alternative c.

Case 3. $p = n - 1$ yields for (5) the relation

$$b(F_{n+1} - F_{n-2}) = a(F_{n-3} + F_{n+2}),$$

which, by some manipulations similar to those in the previous cases and left to the reader, can be transformed into

$$(2b - 3a)F_{n-1} = aF_n,$$

or, since $F_{n-1} \neq 0$,

$$\frac{2b - 3a}{a} = \frac{F_n}{F_{n-1}}.$$

If a is odd, then a and $2b - 3a$ are relatively prime (remark 3); hence, for some even $n > 0$,

$$a = F_{n-1} \quad \text{and} \quad 2b - 3a = F_n,$$

or

$$a = F_{n-1} \quad \text{and} \quad 2(b - a) = F_{n+1},$$

which is alternative d. Finally, if a is even, then $\frac{1}{2}a$ and $b - \frac{3}{2}a$ are relatively prime (remark 4); hence, for some even $n > 0$,

$$\tfrac{1}{2}a = F_{n-1} \quad \text{and} \quad b - \tfrac{3}{2}a = F_n,$$

or

$$\tfrac{1}{2}a = F_{n-1} \quad \text{and} \quad b - a = F_{n+1},$$

which is alternative e.

(ii) As to the converse, the first alternative with $a = 1$ and $b = 3$ generates the Lucas sequence, which is well known to be noninjective, as $L_{-n} = L_n$ for even $n \in N$. The second alternative, with $2a = F_{n-1}$ and $b = F_{n+1}$, where $n \in N$, n even and b odd, generates a sequence H with

$$H_{n+1} = aF_{n-1} + bF_n = 2a^2 + b(b - 2a) = 2a^2 - 2ab + b^2,$$

and

$$H_{-n} = aF_{-n-2} + bF_{-n-1} = -aF_{n+2} + bF_{n+1}$$
$$= -a(2b - 2a) + b^2 = 2a^2 - 2ab + b^2.$$

Hence $H_{-n} = H_{n+1}$ and, since obviously $-n \neq n + 1$, H is noninjective. The third alternative, with $a = F_{n-1}$ and $\frac{1}{2}b = F_{n+1}$, where $n \in N$, n even, b even, and a odd, generates a sequence H with

$$H_{n+1} = aF_{n-1} + bF_n = a^2 + b(\tfrac{1}{2}b - a) = a^2 - ab + \tfrac{1}{2}b^2$$

and

$$H_{-n} = aF_{-n-2} + bF_{-n-1} = -aF_{n+2} + bF_{n+1}$$
$$= -a(b - a) + \tfrac{1}{2}b^2 = a^2 - ab + \tfrac{1}{2}b^2.$$

Hence, again, $H_{-n} = H_{n+1}$ and H is noninjective. The fourth alternative, with $a = F_{n-1}$ and $2(b - a) = F_{n+1}$, where $n \in N$, n is even and a is odd, generates a sequence H with

$$H_{n-1} = aF_{n-3} + bF_{n-2} = a(5a - 2b) + b(2b - 4a) = 5a^2 - 6ab + 2b^2$$

and

$$H_{-n} = aF_{-n-2} + bF_{-n-1} = -aF_{n+2} + bF_{n+1}$$
$$= -a(4b - 5a) + b(2b - 2a) = 5a^2 - 6ab + 2b^2.$$

Hence $H_{-n} = H_{n-1}$ and H is noninjective. Finally, the fifth alternative, with $\frac{1}{2}a = F_{n-1}$ and $b - a = F_{n+1}$, where $n \in N$, n even, a even, and b odd, generates a sequence H with

$$H_{n-1} = aF_{n-3} + bF_{n-2} = a(\tfrac{5}{2}a - b) + b(b - 2a) = \tfrac{5}{2}a^2 - 3ab + b^2$$

and

$$H_{-n} = aF_{-n-2} + bF_{-n-1} = -aF_{n+2} + bF_{n+1}$$
$$= -a(2b - \tfrac{5}{2}a) + b(b - a) = \tfrac{5}{2}a^2 - 3ab + b^2.$$

Hence, again, $H_{-n} = H_{n-1}$ and H is noninjective, which completes the proof.

Examples of noninjective sequences according to the alternatives of Theorem 3 are:

1. The Lucas sequence with characteristic pair (1, 3).
2. The sequence with characteristic pair (1, 5). Here
$$2a = 2 = F_3,\ b = 5 = F_5,\ H_{-4} = H_5 = 17.$$
3. The sequence with characteristic pair (1, 4). Here
$$a = 1 = F_1,\ \tfrac{1}{2}b = 2 = F_3,\ H_{-2} = H_3 = 5.$$

4. The sequence with characteristic pair (13, 30). Here

$$a = 13 = F_7,\ 2(b - a) = 34 = F_9,\ H_{-8} = H_7 = 305.$$

5. The sequence with characteristic pair (4, 9). Here

$$\tfrac{1}{2}a = 2 = F_3,\ (b - a) = 5 = F_5,\ H_{-4} = H_3 = 13.$$

The proof of the following corollary uses the fact that the ratios of successive Fibonacci numbers,

$$\frac{F_n}{F_{n-1}},\ n \in N,\ n > 1,$$

form a sequence which, for $n \to \infty$, converges to

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad (= 1.61803398875\ldots) \quad \text{(see [3], pp. 28, 29).}$$

In particular, the subsequence consisting of the ratios where the numerators have even indices, contains only terms $< \alpha$ and converges to α from below:

$$\frac{1}{1}, \frac{3}{2}, \frac{8}{5}, \frac{21}{13}, \frac{55}{34}, \ldots .$$

This sequence is strictly increasing, i.e., if $\dfrac{F_n}{F_{n-1}} = \dfrac{F_m}{F_{m-1}}$, then $n = m$; moreover,

$$1 \leqslant \frac{F_n}{F_{n-1}} < 1.62.$$

Corollary 1

Let $H_1 = a$, $H_2 = b$, $0 < 2a < b$, a and b relatively prime, and $(a, b) \neq (1, 3)$. Moreover, let H be noninjective. Then there is a *unique* pair $n \in N$ and $p \in N$, where n is even and either $p = n - 1$ or $p = n + 1$, such that $H_{-n} = H_p$.

Proof: The hypothesis that H is noninjective implies that there is a pair $p \in N$ and even $n \in N$ such that $H_{-n} = H_p$. The hypothesis that $(a, b) \neq (1, 3)$ implies that p is either $n + 1$ or $n - 1$. In case $p = n + 1$, the proof of the theorem arrives at

$$\frac{F_n}{F_{n-1}} = \frac{b - 2a}{2a}.$$

Assuming that n, p is not unique, one obtains a different pair, say $q \in N$ and even $m \in N$, such that $H_{-m} = H_q$. If $q = m + 1$, then

$$\frac{F_m}{F_{m-1}} = \frac{b - 2a}{2a}$$

also, and $m = n$ and $q = p$, since equal ratios imply equal indices (see the remarks preceding this corollary), contrary to the assumption that m, q was different from n, p. If $q = m - 1$, then the proof of the theorem yields

$$\frac{F_m}{F_{m-1}} = \frac{2b - 3a}{a} \quad \text{or} \quad \frac{F_m}{F_{m-1}} = 4 \cdot \frac{b - 2a}{2a} + 1 \quad \text{or} \quad \frac{F_m}{F_{m-1}} = 4 \cdot \frac{F_n}{F_{n-1}} + 1.$$

Even if F_n/F_{n-1} is as small as possible, namely $F_2/F_1 = 1$, then, still, $F_m/F_{m-1} = 5$ contrary to the fact that for even m, $F_m/F_{m-1} < 1.62$. Hence, in case $p = n + 1$, the pair n, p is the unique pair such that $H_{-n} = H_p$. In case $p = n - 1$, the argument is the same, be it in reversed order. In this case

$$\frac{F_n}{F_{n-1}} = \frac{2b - 3a}{a}$$

and a different pair, m, q with $q = m - 1$, would also yield

$$\frac{F_m}{F_{m-1}} = \frac{2b - 3a}{a}$$

and $n = m$, contrary to the assumption of different pairs; and a different pair, m, q with $q = m + 1$, would yield

$$\frac{F_m}{F_{m-1}} = \frac{b - 2a}{2a}$$

or

$$\frac{F_m}{F_{m-1}} = \frac{1}{4}\left(\frac{F_n}{F_{n-1}} - 1\right),$$

and since $F_n/F_{n-1} < 1.62$, this would yield $F_m/F_{m-1} < 1$, contrary to the remarks preceding the corollary.

The following table lists the first twenty noninjective sequences, ordered lexicographically by their generating characteristic pairs (a, b), where $0 < 2a < b$ and a and b are relatively prime.

Characteristic Pair	Equal Terms	Alternative of Theorem 3
(1, 3)	$H_{-2n} = H_{2n}$, $n \in N$	1
(1, 4)	$H_{-2} = H_3 = 5$	3
(1, 5)	$H_{-4} = H_5 = 17$	2
(4, 9)	$H_{-4} = H_3 = 13$	5
(5, 26)	$H_{-6} = H_7 = 233$	3
(10, 23)	$H_{-6} = H_5 = 89$	5
(13, 30)	$H_{-8} = H_7 = 305$	4
(13, 68)	$H_{-8} = H_9 = 1597$	3
(17, 89)	$H_{-10} = H_{11} = 5473$	2
(68, 157)	$H_{-10} = H_9 = 4181$	5
(89, 466)	$H_{-12} = H_{13} = 75025$	3
(178, 411)	$H_{-12} = H_{11} = 28657$	5
(233, 538)	$H_{-14} = H_{13} = 98209$	4
(233, 1220)	$H_{-14} = H_{15} = 514229$	3
(305, 1597)	$H_{-16} = H_{17} = 1762289$	2
(1220, 2817)	$H_{-16} = H_{15} = 1346269$	5
(1597, 8362)	$H_{-18} = H_{19} = 24157817$	3
(3194, 7375)	$H_{-18} = H_{17} = 9227465$	5
(4181, 9654)	$H_{-20} = H_{19} = 31622993$	4
(4181, 21892)	$H_{-20} = H_{21} = 165580141$	3

The above table turns out to be considerably more than a list. It suggests several more corollaries to Theorem 3, only one of which will be mentioned here; the proof is left to the reader.

Corollary 2

Every even $n > 2$ determines exactly two ordered pairs of integers, (a, b) and (c, d), with $0 < 2a < b$, $0 < 2c < d$, $b > 3$, $d > 3$, $b \neq d$, a and b relatively prime, c and d relatively prime, and such that the sequence generated by one of the pairs has $H_{-n} = H_{n+1}$ and the sequence generated by the other pair has $H_{-n} = H_{n-1}$.

It should be noticed that $n = 2$ also determines two ordered pairs of integers, (a, b) and (c, d), generating noninjective sequences, but with slight modification that one of the pairs, say (a, b), has $0 < 2a = b$, namely the pair (1, 2) generating the sequence with $H_{-2} = H_1 = 1$ which is F, shifted one place.

Remark: If H is injective, then the terms of H form an abelian group under "multiplication" defined by $H_m H_n = H_{m+n}$, where $m \in Z$, $n \in Z$, with H_0 as multiplicative identity, and $H_n^{-1} = H_{-n}$. See also [2].

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