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ADVANCED PROBLEMS AND SOLUTIONS

Edited by

RAYMOND E. WHITNEY

Lock Haven State College, Lock Haven, PA 17745

Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-352 *Proposed by Stephen Turner, Babson College, Babson Park, Mass.*

One night during a national mathematical society convention, n mathematicians decided to gather in a suite at the convention hotel for an "after hours chat." The people in this group share the habit of wearing the same kind of hats, and each brought his hat to the suite. However, the chat was so engaging that at the end of the evening each (being deep in thought and oblivious to the practical side of matters) simply grabbed a hat at random and carried it away by hand to his room.

Use a variation of the Fibonacci sequence for calculating the probability that none of the mathematicians carried his own hat back to his room.

H-353 *Proposed by Jerry Metzger, Univ. of North Dakota, Grand Forks, ND*

For a positive integer n, describe all two-element sets $\{a, b\}$ for which there is a polynomial $f(x)$ such that $f(x) \equiv 0 \pmod{n}$ has solution set exactly $\{a, b\}$.

H-354 *Proposed by Paul Bruckman, Concord, CA*

Find necessary and sufficient conditions so that a solution in relatively prime integers x and y can exist for the Diophantine equation:

$$ax^2 - by^2 = c,$$

given that a, b, and c are pairwise relatively prime positive integers, and, moreover, a and b are not both perfect squares.

H-355 *Proposed by Gregory Wulczyn, Bucknell Univ., Lewisburg, PA*

Solve the second-order finite difference equation:

$$n(n-1)a_n - \{2rn - r(r+1)\}a_{n-r} + r^2 a_{n-2r} = 0.$$

r and n are integers. If $n - kr < 0$, $a_{n-kr} = 0$.

SOLUTIONS

Al Gebra

H-335 *Proposed by Paul Bruckman, Concord, CA*
(Vol. 20, no. 1, February 1982)

Find the roots, in exact radicals, of the polynomial equation:

$$p(x) = x^5 - 5x^3 + 5x - 1 = 0. \tag{1}$$

Solution by M. Wachtel, Zürich, Switzerland

It is easy to see that one of the solutions is: $x = 1$.

Step 1: Dividing the original equation by $x - 1$, we obtain

$$x^4 + x^3 - 4x^2 - 4x + 1 = 0.$$

Step 2: To eliminate x^3, we set $x = z - \frac{1}{4}$, which yields:

$$z^4 - \frac{35}{8}z^2 - \frac{15}{8}z + \frac{445}{256} = 0.$$

Step 3: Using the formula $t^3 + \frac{p}{2}t^2 + \frac{1}{4}\left[\left(\frac{p}{2}\right)^2 - r\right]t - \left(\frac{q}{8}\right)^2 = 0$, and setting $p = -\frac{35}{8}$, $q = -\frac{15}{8}$, $r = \frac{445}{256}$, the above equation is transformed into a cubic equation:

$$t^3 - \frac{35}{16}t^2 + \frac{195}{256}t - \frac{225}{4096} = 0.$$

Step 4: To eliminate t^2, we set $t = u + \frac{35}{48}$, which yields:

$$u^3 - \frac{5}{6}u - \frac{475}{1728} = 0.$$

Step 5: Using the Cardano formula, we obtain the "Casus irreduzibilus" ($\cos 3\alpha$) with three real solutions:

$$u_1 = -\frac{5}{12}; \quad u_2 = \frac{5 + 9\sqrt{5}}{24}; \quad u_3 = \frac{5 - 9\sqrt{5}}{24};$$

and it follows $\left(t = u + \frac{35}{48}\right)$:

$$t_1 = \frac{5}{16};\quad t_2 = \frac{15 + 6\sqrt{5}}{16};\quad t_3 = \frac{15 - 6\sqrt{5}}{16}.$$

Further:

$$U = \sqrt{t_1} = \frac{\sqrt{5}}{4};\quad V = \sqrt{t_2} = \frac{\sqrt{15 + 6\sqrt{5}}}{4};\quad W = \sqrt{t_3} = \frac{\sqrt{15 - 6\sqrt{5}}}{4}.$$

Step 6: Considering $x = z - \frac{1}{4}$ and the identities

$$\frac{\sqrt{15 + 6\sqrt{5}}}{4} \pm \frac{\sqrt{15 - 6\sqrt{5}}}{4} = \frac{\sqrt{30 \pm 6\sqrt{5}}}{4}$$

we obtain the following solutions:

$$x_0 = 1$$

$$x_1 = U + V + W = \frac{\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1}{4}$$

$$x_2 = U - V - W = \frac{-\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1}{4} \qquad x_1 \cdot x_2 = -\left(\frac{\sqrt{5} + 1}{2}\right)^2$$

$$x_3 = -U + V - W = \frac{\sqrt{30 - 6\sqrt{5}} - \sqrt{5} + 1}{4} \qquad x_3 \cdot x_4 = -\left(\frac{\sqrt{5} - 1}{2}\right)^2$$

$$x_4 = -U - V + W = \frac{-\sqrt{30 - 6\sqrt{5}} - \sqrt{5} + 1}{4} \qquad x_0 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 1$$

The proofs of Steps 3 and 5 are tedious, but the respective formulas can be found in formula registers of algebra.

Also solved by the proposer. (One incorrect solution was received.)

Mod Ern

H-336 (Corrected) *Proposed by Lawrence Somer, Washington, D.C.*
(Vol. 20, no. 1, February 1982)

Let p be an odd prime.

(a) Prove that if $p \equiv 3$ or $7 \pmod{20}$, then

$$5F^2_{(p-1)/2} \equiv -1 \pmod{p} \text{ and } 5F^2_{(p+1)/2} \equiv -4 \pmod{p}.$$

(b) Prove that if $p \equiv 11$ or $19 \pmod{20}$, then

$$5F^2_{(p-1)/2} \equiv 4 \pmod{p} \text{ and } 5F^2_{(p+1)/2} \equiv 1 \pmod{p}.$$

(c) Prove that if $p \equiv 13$ or $17 \pmod{20}$, then

$$F^2_{(p-1)/2} \equiv -1 \pmod{p} \text{ and } F_{(p+1)/2} \equiv 0 \pmod{p}.$$

(d) Prove that if $p \equiv 1$ or $9 \pmod{20}$, then

$$F_{(p-1)/2} \equiv 0 \pmod{p} \text{ and } F_{(p+1)/2} \equiv \pm 1 \pmod{p}.$$

Show that both the cases $F_{(p+1)/2} \equiv -1 \pmod{p}$ and $F_{(p+1)/2} \equiv 1 \pmod{p}$ do in fact occur.

Solution by the proposer

It is known that $F_p \equiv (5/p) \pmod{p}$ and $F_{p-(5/p)} \equiv 0 \pmod{p}$, where $(5/p)$ is the Legendre symbol. It is further known that

$$F_{(\frac{1}{2}(p-(5/p)))} \equiv 0 \pmod{p}$$

if and only if $(-1/p) = 1$. (See [1] or [3]). We also make use of the following identities:

$$F_{2n} = F_n(F_{n-1} + F_{n+1}). \tag{1}$$

$$F_{2n+1} = F^2_n + F^2_{n+1}. \tag{2}$$

Letting $k = (p - 1)/2$, we are now ready to prove parts (a)-(d).

(a) In this case $(5/p) = (-1/p) = -1$. Then, by (1) and (2),

$$F^2_{p+1} = F_{k+1}(F_k + F_{k+2}) \equiv 0 \pmod{p}. \tag{3}$$

and

$$F^2_p = F^2_k + F^2_{k+1} \equiv -1 \pmod{p}. \tag{4}$$

Since $(-1/p) = -1$, $F_k \not\equiv 0 \pmod{p}$. Thus, by (3), $F_{k+2} \equiv -F_k \pmod{p}$. Hence,

$$F_k = F_{k+2} - F_{k+1} \equiv -F_k - F_{k+1} \pmod{p}.$$

Thus, $2F_k \equiv -F_{k+1} \pmod{p}$ and $4F^2_k \equiv F^2_{k+1} \pmod{p}$. Thus, by (4),

$$F^2_p \equiv F^2_k + 4F^2_k = 5F^2_{(p-1)/2} \equiv -1 \pmod{p}.$$

Since $F^2_{k+1} \equiv 4F^2_k$, $5F^2_{(p+1)/2} \equiv 4(5F^2_{(p-1)/2}) \equiv -4 \pmod{p}$.

(b) In this case $(5/p) = 1$ and $(-1/p) = -1$. Then

$$F^2_{p-1} = F_k(F_{k-1} + F_{k+1}) \equiv 0 \pmod{p}$$

and

$$F^2_p = F^2_k + F^2_{k+1} \equiv 1 \pmod{p}.$$

Making use of the fact that $F_k \not\equiv 0 \pmod{p}$ and solving as in the solution of part (a), we find that

$$5F^2_{(p+1)/2} \equiv 1 \pmod p \text{ and } 5F^2_{(p-1)/2} \equiv 4 \pmod p.$$

(c) In this case $(5/p) = -1$ and $(-1/p) = 1$. Thus, $F_k \equiv 0 \pmod p$. Also,

$$F_p^2 = F_k^2 + F_{k+1}^2 \equiv F_{k+1}^2 \equiv -1 \pmod p.$$

(d) In this case $(5/p) = (-1/p) = 1$. Thus, $F_k \equiv 0 \pmod p$. Also,

$$F_p^2 = F_k^2 + F_{k+1}^2 \equiv F_{k+1}^2 \equiv 1 \pmod p.$$

Thus, $F_{k+1} \equiv \pm 1 \pmod p$. For $p = 29$, 89, 101, or 281, $F_{(p+1)/2} \equiv 1 \pmod p$ and for $p = 41$, 61, 109, or 409, $F_{(p+1)/2} \equiv -1 \pmod p$. These examples were obtained from [2].

References

1. Robert P. Backstrom. "On the Determination of the Zeros of the Fibonacci Sequence." *The Fibonacci Quarterly* 4, no. 4 (1966):313-22.
2. *Tables of Fibonacci Entry Points*. Santa Clara, Calif.: The Fibonacci Association, January 1965.
3. D. H. Lehmer. "An Extended Theory of Lucas' Functions." *Annals of Mathematics*, Second Series 31 (1910:419-48.

Also solved by P. Bruckman and G. Wulczyn.

Pivot

H-337 *Proposed by Gregory Wulczyn, Bucknell Univ., Lewisburg, PA (Vol. 20, no. 1, February 1982)*

(a) Evaluate the determinant

$$\det A$$

$$\begin{array}{c} \begin{vmatrix} 1 & -4L_{2r} & 6L_{4r}+16 & -(4L_{6r}+24L_{2r}) & L_{8r}+16L_{4r}+36 \\ L_{2r} & -(3L_{4r}+10) & 3L_{6r}+25L_{2r} & -(L_{8r}+25L_{4r}+60) & 10L_{6r}+60L_{2r} \\ L_{4r} & -(2L_{6r}+6L_{2r}) & L_{8r}+12L_{4r}+30 & -(6L_{6r}+50L_{2r}) & 30L_{4r}+80 \\ L_{6r} & -(L_{8r}+7L_{4r}) & 7L_{6r}+21L_{2r} & -(21L_{4r}+70) & 70L_{2r} \\ L_{8r} & -8L_{6r} & 28L_{4r} & -56L_{2r} & 140 \end{vmatrix} \end{array} \begin{array}{c} (e) \\ (d) \\ (c) \\ (b) \\ (a) \end{array}$$

(b) Show that

$$\begin{aligned} 625F_{2r}^2 &= L_{8r}^2 - 8L_{6r}^2 + 28L_{4r}^2 - 56L_{2r}^2 + 140 \\ &= -8L_{6r}^2 + (L_{8r} + 7L_{4r})^2 - 14(L_{6r} + 3L_{2r})^2 + 7(3L_{4r+10})^2 - 280L_{2r}^2 \\ &= 28L_{4r}^2 - 14(L_{6r} + 3L_{2r})^2 + (L_{8r} + 12L_{4r} + 30)^2 - 2(3L_{6r} + 25L_{2r})^2 \\ &\qquad + 20(3L_{4r} + 8)^2 \\ &= -56L_{2r}^2 + 7(3L_{4r} + 10)^2 - 2(3L_{6r} + 25L_{2r})^2 + (L_{8r} + 25L_{4r} + 60)^2 \\ &\qquad - 40(L_{6r} + 6L_{2r})^2. \end{aligned}$$

Grace Note: If the elements of this determinant are the coefficients of a 5 x 5 linear homogeneous system, then the solution to the 4 x 5 system represented by Equations (b), (c), (d), and (e) is given by the elements of the first column. The solution to (a), (c), (d), and (e) is given by the elements of the second column. And so on.

Solution by the proposer

(a) Using a Chio pivot reduction and taking out $L_{4r} - 2$ as a common factor from each term

$$\text{Det } A = (L_{4r} - 2)^4 \begin{vmatrix} 1 & -3L_{2r} & 3L_{4r} + 9 & -(L_{6r} + 9L_{2r}) \\ 2L_{2r} & -(5L_{4r} + 14) & 4L_{6r} + 26L_{2r} & -(L_{8r} + 18L_{4r} + 42) \\ 3L_{4r} + 3 & -(6L_{6r} + 21L_{2r}) & 4L_{8r} + 32L_{4r} + 63 & -(L_{10r} + 18L_{6r} + 71L_{2r}) \\ 4L_{6r} + 4L_{2r} & -(6L_{8r} + 28L_{4r} + 28) & 4L_{10r} + 32L_{6r} + 84L_{2r} & -(L_{12r} + 18L_{8r} + 71L_{4r} + 140) \end{vmatrix}$$

$$= (L_{4r} - 2)^7 \begin{vmatrix} 1 & -2L_{2r} & L_{4r} + 4 \\ 3L_{2r} & -(5L_{4r} + 14) & 2L_{6r} + 16L_{2r} \\ 6L_{4r} + 8 & -(8L_{6r} + 32L_{2r}) & 3L_{8r} + 28L_{4r} + 58 \end{vmatrix}$$

$$= (L_{4r} - 2)^9 \begin{vmatrix} 1 & -L_{2r} \\ 4L_{2r} & -(3L_{4r} + 10) \end{vmatrix} = (L_{4r} - 2)^{10}$$

$$= (5F_{2r}^2)^{10} = 5^{10}F_{2r}^{20}.$$

(b) $$L_{8r}^2 - 8L_{6r}^2 + 28L_{4r}^2 - 56L_{2r}^2 + 140 = L_{16r} - 8L_{12r} + 28L_{8r} - 56L_{4r} + 70$$
$$= 625F_{2r}^8.$$

$$-8L_{6r}^2 + (L_{8r} + 7L_{4r})^2 - 14(L_{6r} + 3L_{2r})^2 + 7(3L_{4r} + 10)^2 - 280L_{2r}^2$$

$$= L_{16r} + L_{12r}(14 - 8 - 14) + L_{8r}(49 - 84 + 63) + L_{4r}(14 - 84 - 126 + 420 - 280)$$
$$- 16 + 2 + 98 - 28 - 252 + 126 + 700 - 560$$

$$= L_{16r} - 8L_{12r} + 28L_{8r} - 56L_{4r} + 70 = 625F_{2r}^8$$

$$28L_{4r}^2 - 14(L_{6r} + 3L_{2r})^2 + (L_{8r} + 12L_{4r} + 30)^2 - 2(3L_{6r} + 25L_{2r})^2 + 20(3L_{4r} + 8)^2$$

$$= L_{16r} + L_{12r}(-14 + 24 - 18) + L_{8r}(28 - 84 + 144 + 60 - 300 + 180) + L_{4r}(-84 - 126 + 24 + 720 - 300 - 1250 + 960) + 56 - 28 - 252 + 2 + 288 + 900 - 36 - 2500 + 360 + 1280$$

$$= L_{16r} - 8L_{12r} + 28L_{8r} - 56L_{4r} + 70 = 625F_{2r}^8$$

$$-56L_{2r}^2 + 7(3L_{4r} + 10)^2 - 2(3L_{6r} + 25L_{2r})^2 + (L_{8r} + 25L_{4r} + 60)^2 - 40(L_{6r} + 6L_{2r})^2$$

$$= L_{16r} + L_{12r}(-18 + 50 - 40) + L_{8r}(63 - 300 + 120 + 625 - 480) + L_{4r}(-56 + 420 - 300 - 1250 + 50 + 3000 - 480 - 1440) - 112 + 126 + 700 - 36 - 2500 + 2 + 1250 + 3600 - 80 - 2880$$

$$= L_{16r} - 8L_{12r} + 28L_{8r} - 56L_{4r} + 70 = 625F_{2r}^8.$$

Some Abundance

H-338 *Proposed by Charles R. Wall, Trident Tech. Coll., Charleston, SD (Vol. 20, no. 1, February 1982)*

An integer n is abundant if $\sigma(n) > 2n$, where $\sigma(n)$ is the sum of the divisors of n. Show that there is a probability of at least:

(a) 0.15 that a Fibonacci number is abundant;

(b) 0.10 that a Lucas number is abundant.

Solution by the proposer

Three well-known background facts are needed:

1. Any multiple of an abundant number is abundant.

2. F_{nm} is a multiple of F_n for all m.

3. L_{nm} is a multiple of L_n if m is odd.

From published tables of factors of Fibonacci numbers, we see that F_n is abundant if n is 12, 18, 30, 40, 42, 140, 315, 525, or 725. Since none of these numbers is a multiple of any other, the probability that a Fibonacci number is abundant is at least

$$\frac{1}{6}\left(1 - \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\right) + \frac{1}{140}\cdot\frac{2}{3} + \frac{1}{40}\cdot\frac{2}{3}\cdot\frac{6}{7} + \frac{1}{2}\cdot\frac{1}{105}\left(1 - \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\right) = \frac{184}{1255}$$

$$= 0.1502\ldots .$$

Also, L_n is abundant if n is 6, 45, 75, or 105, and so the probability that a Lucas number is abundant is at least

$$\frac{1}{2\cdot 6} + \frac{1}{2\cdot 15}\left(1 - \frac{2}{3}\ \frac{4}{5}\ \frac{6}{7}\right) = \frac{71}{700} = 0.1014\ldots .$$

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