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ON EXPONENTIAL SERIES EXPANSIONS AND CONVOLUTIONS

M. E. COHEN

California State University, Fresno, CA 93740

and

D. L. HUDSON

University of California, Davis, CA 95616

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1. INTRODUCTION

With the aid of the Lagrange Theorem, Pólya and Szegö [10, pp. 301, 302, Problems 210, 214] deduced the very important expansions

$$e^{-z} = \sum_{n=0}^{\infty} \frac{(w)^n (\ell n + 1)^{n-1}}{n!}, \quad w = -ze^{z\ell}, \tag{1.1}$$

and

$$\frac{e^{-z}}{1 + z\ell} = \sum_{n=0}^{\infty} \frac{(w)^n (\ell n + 1)^n}{n!}. \tag{1.2}$$

For applications of the above equations, see Cohen [4], Knuth [8, Section 2.3.4.4], Riordan [12, Section 4.5]. In fact, (1.1) was of interest to Ramanujan [11, p. 332, Question 738]. The higher-dimensional extensions and their ramifications were studied by Carlitz [1], [2], Cohen [5], and others.

A two-dimensional generalization of (1.2) is one result presented in this paper:

For α, λ, a, c real or complex,

$$\sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-x)^p (-y)^k \exp\left[\dfrac{x(\lambda + cp)}{(\alpha + ak)} + \dfrac{y(\alpha + ak)}{(\lambda + cp)}\right] (\alpha + ak)^{k-p} (\lambda + cp)^{p-k}}{p!k!}$$

$$= \frac{1 - \dfrac{acxy}{\alpha\lambda}}{\left(1 + \dfrac{ay}{\lambda}\right)\left(1 + \dfrac{cx}{\alpha}\right)} \tag{1.3}$$

where the double series is assumed convergent.

$x = 0$, along with other appropriate substitutions, reduces (1.3) to (1.2). For other similar two-dimensional exponential series, see Carlitz [2, Equations (1.4) and (1.9)] and Cohen [5, Equation 2.28].

With the aid of (1.3), we obtain a new convolution:

$$\sum_{k=0}^{n}\sum_{p=0}^{m}\frac{\left[-\frac{(\lambda+cp)}{(\alpha+ak)}\right]^{p}\left[\frac{(\lambda+cp)}{(\alpha+ak)}+s\right]^{m-p}\left[-\frac{(\alpha+ak)}{(\lambda+cp)}\right]^{k}\left[\frac{(\alpha+ak)}{(\lambda+cp)}+t\right]^{n-k}}{p!(m-p)!k!(n-k)!}$$
$$=\frac{s^m}{m!}\sum_{j=0}^{n}\frac{(-a/\lambda)^j t^{n-j}}{(n-j)!}-\frac{s^m t^n}{m!n!}+\frac{t^n}{n!}\sum_{i=0}^{m}\frac{(-c/\alpha)^i s^{m-i}}{(m-i)!}. \tag{1.4}$$

(1.4) may be considered as a two-dimensional extension of the Abel-type Gould [7] convolution. See also Carlitz [3] and, for another type of two-dimensional generalization, refer to Cohen [6]. Letting $m = 0$ in (1.4) and simplifying, one obtains the expressions (2) and (4) given in [6]. For an excellent discussion of convolutions, see Riordan [12, Sections 1.5 and 1.6].

A two-dimensional generalization of both (1.1) and (1.2) is also presented here:

For α, λ, μ, a, c, d real or complex,

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-x)^p(-y)^k\exp\left[\frac{x(\lambda+cp)}{(\alpha+ak)}+\frac{y(\alpha+ak)(\mu+dp)}{(\lambda+cp)}\right]}{k!p!}$$
$$\cdot(\alpha+ak)^{k-p}(\lambda+cp)^{p-k-1}(\mu+dp)^k=\frac{1}{(\lambda+ya\mu)}, \tag{1.5}$$

where the double series is assumed convergent.

$y = 0$ and simplification gives (1.1), and $x = 0$ and reduction yields (1.2).

(1.5) is employed in the proof of the new expression:

$$\sum_{k=0}^{n}\sum_{p=0}^{m}\frac{\left[\frac{-(\lambda+cp)}{(\alpha+ak)}\right]^{p}\left[\frac{(\lambda+cp)}{(\alpha+ak)}+s\right]^{m-p}\left[\frac{-(\alpha+ak)(\mu+dp)}{(\lambda+cp)}\right]^{k}\left[\frac{(\alpha+ak)(\mu+dp)}{(\lambda+cp)}+t\right]^{n-k}}{p!(m-p)!k!(n-k)!(\lambda+cp)}$$
$$=\frac{s^m}{\lambda m!}\sum_{j=0}^{n}\frac{(-a\mu/\lambda)^j t^{n-j}}{(n-j)!}. \tag{1.6}$$

(1.6) may be regarded as a two-dimensional extension of the Abel-type Gould convolution to which it reduces for $m = 0$.

Another generalization of (1.1) is the expression,

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-x)^p(-y)^k}{k!p!}\exp\left\{\frac{x(\beta+bk)(\lambda+cp)}{(\alpha+ak)}+\frac{y(\alpha+ak)(\mu+dp)}{(\lambda+cp)}\right\}$$

$$\cdot\,(\alpha+ak)^{k-p-1}(\beta+bk)^p(\lambda+cp)^{p-k-1}(\mu+dp)^k$$

$$=\frac{1}{\alpha\lambda}\,{}_1F_2\left[1;\ (\alpha/a)+1,\ (\lambda/c)+1;\ \left(\frac{\alpha b}{a}-\beta\right)\left(\frac{\lambda d}{c}-\mu\right)xy\right], \tag{1.7}$$

where α, β, λ, μ, a, b, c, d are real or complex, and the double series is assumed to be convergent.

The ${}_1F_2$ hypergeometric function is defined in Luke [9, p. 155]. In fact, this particular function is called the "Lommel function," given by [9, p. 413, Equation 1]. Letting $x = 0$ in (1.7) gives (1.1).

With the aid of (1.7), we are able to prove the expansion,

$$\sum_{k=0}^{n}\sum_{p=0}^{m}\frac{\left(\left[\frac{-(\beta+bk)(\lambda+cp)}{(\alpha+ak)}\right]^p\left[\frac{(\beta+bk)(\lambda+cp)}{(\alpha+ak)}+s\right]^{m-p}\cdot\left[\frac{-(\alpha+ak)(\mu+dp)}{(\lambda+cp)}\right]^k\left[\frac{(\alpha+ak)(\mu+dp)}{(\lambda+cp)}+t\right]^{n-k}\right)}{p!(m-p)!k!(n-k)!(\alpha+ak)(\lambda+cp)}$$

$$=\frac{1}{\alpha\lambda}\sum_{i=0}^{\min(m,n)}\frac{s^{m-i}t^{n-i}\left(\frac{\alpha b}{a}-\beta\right)^i\left(\frac{\lambda d}{c}-\mu\right)^i}{(m-i)!(n-i)!\left(\frac{\alpha}{a}+1\right)_i\left(\frac{\lambda}{c}+1\right)_i}, \tag{1.8}$$

where $(a)_n = (a)(a+1)\cdots(a+n-1)$ for $n > 0$,
$\quad = 1$ for $n = 0$.

The proofs of Equations (1.3) through (1.8) are given in the following section.

2. PROOFS OF EQUATIONS (1.3) THROUGH (1.8)

Proof of (1.3)

Consider the expression

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^m y^n}{m!n!}(xD)^{n-m}[x^{\alpha}(1-x^a)^n](xD)^{m-n}[x^{\lambda}(1-x^c)^m]. \tag{2.1}$$

At $x = 1$, it may be expanded to give

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x^m y^n}{m!n!} \sum_{k=0}^{n} \sum_{p=0}^{m} \frac{(-n)_k (\alpha + ak)^{n-m} (-m)_p (\lambda + cp)^{m-n}}{k!p!} \tag{2.2}$$

$$= \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-k)^p (-y)^k}{p!k!} \exp\left[\frac{x(\lambda + cp)}{(\alpha + ak)} + \frac{y(\alpha + ak)}{(\lambda + cp)}\right] (\alpha + ak)^{k-p} (\lambda + cp)^{p-k}. \tag{2.3}$$

The double series transformation,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} f(n, k) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f(n + k, k) \text{ and } (-n)_k = \frac{(-1)^k n!}{(n - k)!}, \tag{2.4}$$

is used over k, n and p, m in going from (2.2) to (2.3). Also, after employing the transformation, the series over m and n are summed to give the exponentials.

Returning to (2.1), it may be observed that the only contributions in that expression give

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{k!} (\alpha + ak)^n \lambda^{-n} + \sum_{m=1}^{\infty} x^m \alpha^m (-c)^m \tag{2.5}$$

$$= \frac{1 - \dfrac{acxy}{\alpha\lambda}}{\left(1 + \dfrac{ay}{\lambda}\right)\left(1 + \dfrac{cx}{\alpha}\right)} \tag{2.6}$$

(2.5) reduces to (2.6) with the aid of (2.4) and series simplification. Equating (2.3) and (2.6) gives the result (1.3).

Proof of (1.4)

Assuming (1.3), multiply both sides of that equation by $\exp[sx + ty]$. The exponentials may be expanded, and the left-hand side assumes the form

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^p (-1)^k x^p y^k (\alpha + ak)^{k-p} (\lambda + cp)^{p-k}}{p!k!}$$
$$\frac{x^m}{m!}\left[\frac{(\lambda + cp)}{(\alpha + ak)} + s\right]^m \frac{y^n}{n!}\left[\frac{(\alpha + ak)}{(\lambda + cp)} + t\right]^n. \tag{2.7}$$

The right-hand side may be expanded to give

$$\left(1 - \frac{ac}{\alpha\lambda}xy\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m y^n \cdot \left\{\frac{s^m}{m!} \sum_{j=0}^{n} \frac{(-a/\lambda)^j t^{n-j}}{(n - j)!} - \frac{s^m t^n}{m!n!} + \frac{t^n}{n!} \sum_{i=0}^{m} \frac{(-c/\alpha)^i s^{m-i}}{(m - i)!}\right\}. \tag{2.8}$$

(2.7) may be expressed as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x^m y^n \sum_{k=0}^{n}\sum_{p=0}^{m} \frac{\left[\frac{-(\lambda+cp)}{(\alpha+ak)}\right]^p \left[\frac{-(\alpha+ak)}{(\lambda+cp)}\right]^k \left[\frac{(\lambda+cp)}{(\alpha+ak)}+s\right]^{m-p} \left[\frac{(\alpha+ak)}{(\lambda+cp)}+t\right]^{n-k}}{p!k!(m-p)!(n-k)!} \tag{2.9}$$

Comparing coefficients between Equations (2.8) and (2.9) gives the result (1.4).

Proof of (1.5)

Consider the expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^m y^n}{m!n!}(xD)^{n-m}[x^{\alpha}(1-x^a)^n](y\delta)^n\left[x^{-\lambda+\frac{\mu c}{d}}(xD)^{m-n-1}[x^{\lambda}(1-x^c)^m]\right], \tag{2.10}$$

where $y = x^{c/d}$, $D \equiv \dfrac{d}{dx}$, $\delta \equiv \dfrac{d}{dy}$.

Following the procedure adopted in the proof of (1.3), (2.10) assumes the form

$$\sum_{p=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-x)^p(-y)^k}{p!k!} \exp\left[\frac{x(\lambda+cp)}{(\alpha+ak)} + \frac{y(\alpha+ak)(\mu+dp)}{(\lambda+cp)}\right]$$
$$\cdot\ (\alpha+ak)^{k-p}(\lambda+cp)^{p-k-1}(\mu+dp)^k. \tag{2.11}$$

Referring to (2.10), it may be seen that at $x = 1$ for $n \geqslant m$, only $m = 0$ contributes and for $n < m$, the expression is zero. Hence, we have

$$\frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{-ay\mu}{\lambda}\right)^n. \tag{2.12}$$

Equating (2.11) and (2.12) gives the result (1.5).

Proof of (1.6)

Following the procedure given in the proof of (1.4), the left-hand side of (1.5) multiplied by $\exp[sx+ty]$ may be expanded as

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^p(-1)^k x^p y^k}{p!k!}(\alpha+ak)^{k-p}(\lambda+cp)^{p-k-1}(\mu+dp)^k$$
$$\frac{x^m}{m!}\left[\frac{(\lambda+cp)}{(\alpha+ak)}+s\right]^m \frac{y^n}{n!}\left[\frac{(\alpha+ak)(\mu+dp)}{(\lambda+cp)}+t\right]^n. \tag{2.13}$$

The right-hand side reduces to

$$\frac{1}{\lambda}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^m y^n s^m}{m!}\sum_{j=0}^{n}\frac{(-a\mu/\lambda)^j t^{n-j}}{(n-j)!}$$

Equating coefficients in (2.13) and (2.14) gives Equation (1.6).

Proof of (1.7)

Consider the operators

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{x^m y^n}{m!n!}(y_1\delta_1)^n\left[x^{-\lambda+\frac{\mu c}{d}}(xD)^{m-n-1}[x^{\lambda}(1-x^c)^m]\right]$$

$$\cdot\ (y_2\delta_2)^m\left[x^{-\alpha+\frac{\beta a}{b}}(xD)^{n-m-1}[x^{\alpha}(1-x^a)^n]\right], \tag{2.15}$$

where $y_1 = x^{c/d}$, $y_2 = x^{a/b}$, $D \equiv \dfrac{d}{dx}$, $\delta_1 \equiv \dfrac{d}{dy_1}$, $\delta_2 \equiv \dfrac{d}{dy_2}$.

As in the proof of (1.3) and (1.5), (2.15) reduces to

$$\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-x)^p(-y)^k}{p!k!}\exp\left\{\frac{x(\beta+bk)(\lambda+cp)}{(\alpha+ak)}+\frac{y(\alpha+ak)(\mu+dp)}{(\lambda+cp)}\right\}$$

$$\cdot\ (\alpha+ak)^{k-p-1}(\beta+bk)^p(\lambda+cp)^{p-k-1}(\mu+dp)^k. \tag{2.16}$$

Now, looking at (2.15), at $x = 1$, and noting that

$$\sum_{k=0}^{n}\frac{(-n)_k(\beta+bk)^n}{k!(\lambda+ck)} = \frac{n!\left(\beta-\frac{\lambda b}{c}\right)^n\Gamma\left(\frac{\lambda}{c}\right)}{c\,\Gamma\left(\frac{\lambda}{c}+n+1\right)} \tag{2.17}$$

with the only contributions coming from $m = n$, one has the reduced expression

$$\frac{1}{\alpha\lambda}\sum_{n=0}^{\infty}\frac{(xy)^n\left(\frac{\alpha b}{a}-\beta\right)^n\left(\frac{\lambda d}{c}-\mu\right)^n}{\left(\frac{\alpha}{a}+1\right)_n\left(\frac{\lambda}{c}+1\right)_n} \tag{2.18}$$

Comparing (2.16) and (2.18) gives (1.7).

Proof of (1.8)

Assuming the expansion (1.7) and following the type of proof adopted for (1.6), with suitable modifications, Equation (1.8) is obtained.

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