KRONECKER'S THEOREM AND RATIONAL APPROXIMATION OF ALGEBRAIC NUMBERS

WAYNE M. LAWTON

Jet Propulsion Laboratory, Pasadena, CA 91109
(Submitted March 1982)

Let the circle T be parametrized by the real numbers modulo the integers. When a real number is used to denote a point in T, it is implied that the fractional part of the number is being considered. If $a, b \in T$ with $a - b \neq .5$, then (a, b) will denote the shortest open arc in T whose endpoints are a and b.

Fix an irrational number x. For any positive integer n let S_n denote the set of n open arcs in T formed by removing the points $x, \ldots, nx$ from T, and let L_n be the length of the longest arc in S_n. Then, the result of Kronecker in [1, p. 363, Theorem 438] implies that $L_n \to 0$ as $n \to \infty$. Without further restrictions on x it is not possible to characterize the rate of convergence of L_n. However, if x is an algebraic number of degree d (that is, if x satisfies a polynomial equation having degree d and integer coefficients), then the following result gives an upper bound for the rate of convergence of L_n.

Theorem 1

If x is an irrational algebraic number of degree d, there exists $c(x) > 0$ such that for all $n > 3$

$$L_n < c(x)/n^{1/(d-1)}. \tag{1}$$

The proof of this theorem is based on the following three lemmas.

Lemma 1

If x is an irrational algebraic number of degree d, there exists $k(x) > 0$ such that, if $(x, px + x)$ is an arc in S_n, then

$$\text{Length } (x, px + x) > k(x)/p^{(d-1)}. \tag{2}$$

Proof: This inequality follows from Liouville's theorem [1, p. 160, Theorem 191].

Lemma 2

If x is irrational and $n > 3$, choose $p < q$ such that $(x,\ px + x)$ and $(x,\ qx + x)$ are arcs in S_n. Then the set S_n can be partitioned into two or three subsets as follows:

$$A_p = \{(kx,\ px + kx)\}:\ 1 \leqslant k \leqslant n - p \tag{3}$$

$$A_q = \{(kx,\ qx + kx)\}:\ 1 \leqslant k \leqslant n - q \tag{4}$$

$$A_r = \{(nx - qx + kx,\ nx - px + kx)\}:\ 1 \leqslant k \leqslant p + q - n. \tag{5}$$

Proof: Let $(a,\ b)$ be any arc in S_n with $a < b$. Then $(a + x,\ b + x)$ is an arc in S_n or $b = nx$ or $(a + x,\ b + x)$ contains the point x. In the latter case, $a = px$ and $b = qx$. Hence, letting $a = x$, $b = px + x$, and successively translating the arc (a, b) by x yields the $n - p$ arcs in set A_p. Similarly, set A_q is formed if $a = x$ and $b = qx + x$. Finally, if $(a,\ b)$ is an arc not contained in A_p or A_q, then successive translation by x must terminate at the arc $(px,\ qx)$. Since there are

$$n - (n - p) - (n - q) = p + q - n$$

arcs in S_n that are not in A_p or in A_q, the proof is complete.

Lemma 3

Assume the hypothesis and notation of Lemma 2. Let I_p and I_q denote the lengths of the arcs in sets A_p and A_q, respectively. Then the arcs in set A_r have length $I_r = I_p + I_q$. Furthermore, the following relations are valid:

$$p + q \geqslant n; \tag{6}$$

$$pI_q + qI_p = 1. \tag{7}$$

Proof: Clearly $I_r = I_p + I_q$, since

$$\begin{aligned} I_r &= \text{length } (px,\ qx) = \text{length } (px + x,\ qx + x) \\ &= \text{length } (px + x,\ x) + \text{length } (x,\ qx + x) \\ &= I_p + I_q. \end{aligned}$$

Also, since the total number of arcs in A_p and A_q does not exceed n,

$$(n - p) + (n - q) \leqslant n;$$

hence, $p + q \geqslant n$, which is inequality (6). Finally, since the sum of the lengths of the arcs in S_n is 1, it follows that

$$1 = (n - p)I_p + (n - q)I_q + (p + q - n)(I_p + I_q) = pI_q + qI_p,$$

which is equality (7). The proof is finished.

Proof of Theorem 1: Assume x is an irrational algebraic number of degree d and that $k(x) > 0$ is chosen as in Lemma 1 so that inequality (2) is valid. Then, for any $n > 3$, choose $p < q$ as in Lemma 2. Therefore, combining inequality (2) with equality (7) yields the following inequality:

$$1 > k(x)[p/q^{d-1} + q/p^{d-1}] > k(x)q/p^{d-1}. \tag{8}$$

This combines with inequality (6) to yield

$$p^{d-1} > k(x)q \geqslant k(x)(n - p). \tag{9}$$

Therefore,

$$p^{d-1} + k(x)p > k(x)n. \tag{10}$$

Clearly, there exists a number $g(x) > 0$ which depends only on $k(x)$ and d such that for every $n > 3$

$$p > g(x)n^{1/d-1}. \tag{11}$$

Substituting inequality (11) into equation (7) yields

$$1 = pI_q + qI_p > p(I_q + I_p) > g(x)n^{1/(d-1)}I_r. \tag{12}$$

Since $L_n \leqslant I_r$, if $c(x) = 1/g(x)$, then inequality (12) implies inequality (1). This completes the proof of Theorem 1.

If in Lemma 1, $d = 2$ and x is irrational and satisfies the equation $ax^2 + bx + c = 0$ and $k(x) < (b^2 - 4ac)^{-1/2}$, then inequality (2) is valid for all except a finite number of values for p.

Clearly, as $n \to \infty$, both $p \to \infty$ and $q \to \infty$; hence, it follows from inequality (10) that inequality (11) is valid for all except a finite number of values for n if

$$g(x) = k(x)/(1 + k(x)).$$

Hence, the inequality (1) in Theorem 1 is valid for all except a finite number of values for n if

$$c(x) = 1/g(x) = 1 + 1/k(x) > 1 + (b^2 - 4ac)^{1/2}.$$

The smallest value of the right side of this inequality occurs for $a = 1$, $b = -1$, $c = -1$ in which case $x = (1 + \sqrt{5})/2$ (the classical "golden ratio"), or $x = (1 - \sqrt{5})/2$.

Remark

The referee has noted that, for algebraic numbers of degree three or more, the bound in Theorem 1 is not the best possible. If Roth's theorem [2, p. 104] is used in place of Liouville's in Lemma 1, then one obtains a bound of the form

$$L_n < c(\varepsilon)/n^{1-\varepsilon} \tag{13}$$

for any $\varepsilon > 0$, where $c(\varepsilon)$ is a constant depending on ε.

ACKNOWLEDGMENT

The research described in this paper was carried out by the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.

REFERENCES

1. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. London: Oxford University Press, 1938.
2. J. W. S. Cassels. *An Introduction to Diophantine Approximation*. London: Cambridge University Press, 1957.

◆◇◆◇◆