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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be submitted on a separate signed sheet, or sheets. Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-502 *Proposed by Herta T. Freitag, Roanoke, VA*

Given that h and k are integers with $h + k$ an integral multiple of 3, prove that $F_k F_{k-h-1} + F_{k+1} F_{k-h}$ is even.

B-503 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Prove that every even perfect number except 28 is congruent to 1 or -1 modulo 7.

B-504 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Prove that if n is an odd integer and F_n is in the set

$$\{0, 1, 3, 6, 10, \ldots\}$$

of triangular numbers, then $n \equiv \pm 1 \pmod{24}$.

B-505 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$N = N(m, a) = L_{m-2a} L_m - L_{m+1-2a} L_{m-1},$$

where m and a are positive integers. Prove or disprove that N is always (exactly) divisible by 5; never divisible by 3, 4, 6, 7, 8, 9, or 11; and is divisible by 10 only if $a \equiv 2 \pmod 3$.

B-506 *Proposed by Heinz-Jürgen Sieffert, student, Berlin, Germany*

Let $G_n = (n + 1)F_n$ and $H_n = (n + 1)L_n$. Prove that:

(a) $$\sum_{k=0}^{n} G_k G_{n-k} = \frac{(n + 2)(n + 3)}{30} H_n - \frac{2}{25} H_{n+2} + \frac{4}{25} F_{n+3};$$

(b) $$\sum_{k=0}^{n} H_k H_{n-k} = \frac{(n + 2)(n + 3)}{6} H_n + \frac{2}{5} H_{n+2} - \frac{4}{5} F_{n+3}.$$

B-507 *Proposed by Heinz-Jürgen Sieffert, Berlin, Germany*

Let G_n and H_n be as in B-506. Find a formula for $\sum_{k=0}^{n} G_k H_{n-k}$ similar to the formulas in B-506.

SOLUTIONS

Fibonacci Norm Identity

B-478 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

(a) Show that the congruence

$$x^2 \equiv -1 \pmod{4m^2 + 4m + 5}$$

has $x = \pm(2m^2 + m + 2)$ as a solution for m in $N = \{0, 1, \ldots\}$.

(b) Show that the congruence

$$x^2 \equiv -1 \pmod{100m^2 + 156m + 61}$$

has a solution $x = am^2 + bm + c$ with fixed integers a, b, c for m in N.

Solution by Paul S. Bruckman, Carmichael, CA

The identities

(1) $$(2m^2 + m + 2)^2 + 1 = (m^2 + 1)(4m^2 + 4m + 5), \text{ and}$$

(2) $$(50m^2 + 53m + 11)^2 + 1 = (25m^2 + 14m + 2)(100m^2 + 156m + 61)$$

are particular instances of the more general identity (due to Fibonacci himself!)

(3) $$(pq - rs)^2 + (ps + qr)^2 = (p^2 + r^2)(q^2 + s^2).$$

Setting $p = 2m + 1$, $q = 1$, $r = 2$, and $s = m$ in (3) yields (1). Setting

$p = 3m + 1$, $q = 8m + 6$, $r = 4m + 1$, and $s = 6m + 5$ in (3) yields (2). This establishes parts (a) and (b) of the problem; in part (b), we have $a = 50$, $b = 53$, $c = 11$.

Also solved by Herta T. Freitag, L. Kuipers, Bob Prielipp, Sahib Singh, J. Suck, M. Wachtel, and the proposer.

Divisibility from a Lucas Sum

B-479 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that $L_{a+nd} + L_{a+nd-d} - L_{a+d} - L_a$ is an integral multiple of L_d for positive integers a, d, and n with d odd.

Solution by J. Suck, Essen, Germany

For positive integers a, d, and n with d odd, we have

$$L_{a+nd} + L_{a+nd-d} - L_{a+d} - L_a = (L_{a+d} + L_{a+2d} + \cdots + L_{a+(n-1)d})L_d.$$

Proof by induction on n: For $n = 1$, both sides equal 0 (the empty sum on the right-hand side). For the step $n \to n + 1$, we have to show that

$$L_{a+(n+1)d} = L_{a+(n-1)d} + L_{a+nd}L_d.$$

This is clear from the identities I_8 and I_{23} of Hoggatt's list; namely,

$$L_k = F_{k-1} + F_{k+1} \quad \text{and} \quad F_{k+p} - F_{k-p} = F_k L_p \text{ for } p \text{ odd}.$$

Thus,

$$\begin{aligned} L_{a+(n-1)d} + L_{a+nd}L_d &= F_{a+(n-1)d-1} + F_{a+(n-1)d+1} + F_{a+(n+1)d-1} \\ &\quad - F_{a+(n-1)d-1} + F_{a+(n+1)d+1} - F_{a+(n-1)d+1} \\ &= L_{a+(n+1)d}. \end{aligned}$$

Also solved by Paul S. Bruckman, Walther Janous, L. Kuipers, Bob Prielipp, Sahib Singh, and the proposer.

Even Case

B-480 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that $L_{a+nd} - L_{a+nd-d} - L_{a+d} + L_a$ is an integral multiple of $L_d - 2$ for positive integers a, d, and n with d even.

Solution by Sahib Singh, Clarion State College, Clarion, PA

This result is true. The proof is based on applying induction on n. The result is obvious when $n = 1$. For $n = 2$, with d even, it is easy to verify that

$$L_{a+2d} - 2L_{a+d} + L_a = (L_d - 2)L_{a+d}.$$

Using the pattern for $n = 3$, we assume the validity of the result:

$$L_{a+nd} - L_{a+nd-d} - L_{a+d} + L_a = (L_{a+d} + L_{a+2d} + \cdots + L_{a+(n-1)d})(L_d - 2).$$

Also, with d even, we have

$$L_{a+(n+1)d} - 2L_{a+nd} + L_{a+(n-1)d} = L_{a+nd}(L_d - 2).$$

By addition, we get the confirmation that the result is true for $(n + 1)$. Thus, the proof is complete.

Also solved by Paul S. Bruckman, Walther Janous, L. Kuipers, Bob Prielipp, J. Suck, and the proposer.

Matching Pennies

B-481 *Proposed by Jerry Metzger, Univ. of North Dakota, Grand Forks, ND*

A and B compare pennies with A winning when there is a match. During an unusual sequence of mn comparisons, A produced m heads followed by m tails followed by m heads, etc., while B produced n heads followed by n tails followed by n heads, etc. By how much did A's wins exceed his losses? (For example, with $m = 3$ and $n = 5$, one has

A: HHHTTTHHHTTTHHH
B: HHHHHTTTTTHHHHH

and A's 8 wins exceed his 7 losses by 1.)

Solution by J. Suck, Essen, Germany

Let $d = \gcd(m, n)$. The excess is 0 if m/d or n/d is even, and d^2 if both are odd.

Proof: The whole double sequence can be split into d blocks of length $mn/d = \operatorname{lcm}(m, n)$. If m/d or n/d is even, then the first block ends in a non-match. Let there be M matches and N non-matches in the first block. Interchanging heads and tails in the row which ends with a T means writing down the block in reverse order, leaving the number of matches unaffected. But matches have become non-matches and vice versa, so that $M = N$. Interchanging heads and tails in both rows of the second block (if $d > 1$) produces the first block in reverse order, and so, again, there is no excess of matches over non-matches. The third block is equal to the first, etc.

If m/d and n/d are odd, odd-numbered blocks are identical, even-numbered blocks equal the first block when heads and tails are interchanged. Now, we may assume that m and n are relatively prime: replacing each symbol by a run of d of the same sort produces the first block of the general case. Furthermore, assume $m < n$.

Insert a bar after every n symbols of the double sequence. Let a_μ resp. $-a_\mu$ (b_μ resp. $-b_\mu$) be the length of the string of successive matches resp.

non-matches preceding (following) the μth bar, $\mu = 1, \ldots, m - 1$. For example, with $m = 3$ and $n = 5$, one has $a_1 = -2$, $b_1 = 1$, $a_2 = 1$, and $b_2 = -2$. Since $d = 1$, $0 < |a_\mu|, |b_\mu| < m$. We have $|a_\mu| + |b_\mu| = m$, and so the excess of matches over non-matches is

$$e := \sum_{\nu=1}^{n-(m-1)} (-1)^{\nu+1}m + \sum_{\mu=1}^{m-1} a_\mu + \sum_{\mu=1}^{m-1} b_\mu.$$

Now, $|a_1|$ won't recur among $|a_2|, \ldots, |a_{m-1}|$ since $d = 1$. Also, $|b_1|$ won't recur among $|b_2|, \ldots, |b_{m-1}|$ since, otherwise, $|a_1|$ would have had to recur, etc. Thus, $|a_1|, \ldots, |a_{m-1}|$ and $|b_1|, \ldots, |b_{m-1}|$ are permutations of $1, \ldots, m - 1$. Setting $b_0 := 0$, let q_μ be defined by

$$n - |b_{\mu-1}| = q_\mu m + |a_\mu|, \quad \mu = 1, \ldots, m - 1.$$

It is clear that

$$\text{if } b_{\mu-1} \text{ is } \begin{cases} \text{even}, < 0 \\ \text{even}, < 0 \\ \text{odd}, \;\; > 0 \\ \text{odd}, \;\; > 0 \end{cases} \text{and } q_\mu \text{ is } \begin{cases} \text{even} \\ \text{odd} \\ \text{even} \\ \text{odd} \end{cases}, \text{ then } a_\mu \text{ is } \begin{cases} \text{odd}, \;\; > 0 \\ \text{even}, < 0 \\ \text{even}, < 0 \\ \text{odd}, \;\; > 0 \end{cases}$$

and that

$$\text{if } a_\mu \text{ is } \begin{cases} \text{even}, < 0 \\ \text{odd}, \;\; > 0 \end{cases}, \text{ then } b_\mu \text{ is } \begin{cases} \text{odd}, \;\; > 0 \\ \text{even}, < 0 \end{cases}.$$

This implies that $e = m + (1 - 2 + 3 - \cdots - (m-1))2 = 1$, which had to be shown.

Also solved by Paul S. Bruckman and the proposer.

Distinct Limits

B-482 *Proposed by John Hughes and Jeff Shallit, Univ. of California, Berkeley, CA*

Find an infinite sequence $a_1, a_2, \ldots$ of positive integers such that

$$\lim_{n\to\infty} (a_n)^{1/n} \quad \text{and} \quad \lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}(a_{k+1}/a_k)\right]$$

both exist but are unequal.

In the following tabulation of the solutions,

$$L = \lim_{n\to\infty}(a_n)^{1/n} \quad \text{and} \quad L' = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}(a_{k+1}/a_k)\right].$$

SOLVER:	a_{2m-1}	a_{2m}	L	L'
Paul S. Bruckman *Carmichael, CA*	1	3	1	5/3
Walther Janous *Univ. Innsbruck, Austria*	1	2	1	5/4
L. Kuipers, *Sierre, Switzerland*	2	4	1	5/4
J. Suck *Essen, Germany*	$a/a^{1/(2m-1)}$	$a \cdot a^{1/(2m)}$	a	$\frac{1}{2}\left(a^3 + \frac{1}{a}\right)$
S. Uchiyama *Univ. of Tsukuba, Japan*	2^{m-1}	2^m	$\sqrt{2}$	3/2
Proposers *Hughes & Shallit*	Same as that of S. Uchiyama			

Limit, No Limit

B-483 *Proposed by John Hughes and Jeff Shallit*
Univ. of California, Berkeley, CA

Find an infinite sequence $a_1, a_2, \ldots$ of positive integers such that

$$\lim_{n\to\infty}(a_n)^{1/n} \text{ exists and } \lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}(a_{k+1}/a_k)\right] \text{ does not exist.}$$

Solution by Walther Janous, Universitaet Innsbruck, Innsbruck, Austria

Let $a_{2m-1} = 1$ and $a_{2m} = m$. Then $\lim(a_n)^{1/n} = 1$. Also

$$\{a_{n+1}/a_n\} = 1,\ 1,\ 2,\ 1/2,\ 3,\ 1/3,\ 4,\ 1/4,\ \ldots,$$

and thus

$$\frac{1}{n}\sum_{k=1}^{n}(a_{k+1}/a_k) > (1 + 2 + 3 + \cdots + [n/2])/n \to \infty \text{ as } n \to \infty.$$

Also solved by Paul S. Bruckman, L. Kuipers, J. Suck, S. Uchiyama, and the proposers.

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