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LUCAS TRIANGLE

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1. INTRODUCTION

Throughout this paper we let $\{F_n\}$ denote the Fibonacci sequence as defined in [1] by

$$F_n = F_{n-1} + F_{n-2},\ F_0 = 1,\ F_1 = 1$$

and $\{L_n\}$ denote the Lucas sequence which is defined by

$$L_n = L_{n-1} + L_{n-2},\ L_0 = 2,\ L_1 = 1.$$

Furthermore, as in [2], we define the set of integers $\{g_{m,n}\}$ by the two relations

$$g_{m,n} = g_{m-1,n} + g_{m-2,n} \qquad (m \geqslant 2,\ m \geqslant n \geqslant 0) \tag{1}$$

$$g_{m,n} = g_{m-1,n-1} + g_{m-2,n-2} \tag{2}$$

where $g_{0,0} = 2$, $g_{1,0} = 1$, $g_{1,1} = 1$, and $g_{2,1} = 2$.

When we arrange this sequence in triangular form, like that of Pascal's triangle, we obtain what shall be called the Lucas triangle where the numbers in the same row have the same index m and as we go from left to right the index n changes from zero to m. See Figure 1.

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                        2
                     1     1
                  3     2     3
               4     3     3     4
            7     5     6     5     7
        11     8     9     9     8    11
      18    13    15    14    15    13    18
    29    21    24    23    23    24    21    29
  47    34    39    39    38    37    39    34    37
76    55    63    60    61    61    60    63    55    76
123   89   102   97    99    98    99    97   102    89   123
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                        .
                        .
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FIGURE 1. LUCAS TRIANGLE

2. CHARACTERISTICS OF THE LUCAS TRIANGLE

Examining the Lucas triangle associated with $\{g_{m,n}\}$, we see that there are two Lucas sequences and two Fibonacci sequences in the triangle:

$$g_{m,0} = g_{m,m} = L_m, \qquad m \geqslant 0 \tag{3}$$

and

$$g_{m,1} = g_{m,m-1} = F_m, \qquad m \geqslant 1. \tag{4}$$

In other words, the first and second "roofs" of the triangle are formed by the familiar sequences $\{F_n\}$ and $\{L_n\}$. Because of the recursive definition for the set $\{g_{m,n}\}$, it is obvious that

$$g_{m,n} = F_{k+1} \cdot g_{m-k,n} + F_k \cdot g_{m-k-1,n} \quad \text{for any } 1 \leqslant k \leqslant m-n-1$$

and

$$g_{m,n} = F_{k+1} \cdot g_{m-k,n-k} + F_k \cdot g_{m-k-1,n-k-1} \text{ for any } 1 \leqslant k \leqslant n-1.$$

Furthermore, the Lucas triangle is symmetrical. That is,

$$g_{m,n} = g_{m,m-n}.$$

In forming the Lucas triangle, we used the following four numbers

$$\{g_{0,0},\ g_{1,0},\ g_{1,1},\ g_{2,1}\}$$

Because of the recursive relations defining $\{g_{m,n}\}$, it is obvious that we could start with any four numbers

$$\{g_{m,n},\ g_{m-1,n},\ g_{m-1,n-1},\ g_{m-2,n-1}\}$$

and, by using (1) and (2), working forward as well as backward, obtain the entire Lucas triangle.

There are also many relations that we could establish for the Lucas triangle, as was done for the Fibonacci triangle in [2]. We mention only a few, since they are so similar in form. First, note that

$$g_{m+2,n+1} = g_{m,n} + g_{m-1,n} + g_{m-1,n-1} + g_{m-2,n-1},$$

$$g_{m-4,n-2} = g_{m,n} - g_{m-1,n} - g_{m-1,n-1} + g_{m-2,n-1},$$

$$g_{m-1,n-2} = g_{m,n} + g_{m-1,n} - g_{m-1,n-1} - g_{m-2,n-1},$$

and

$$g_{m-1,n+1} = g_{m,n} - g_{m-1,n} + g_{m-1,n-1} - g_{m-2,n-1}.$$

Next, consider the three numbers

$$\{g_{m,n},\ g_{m-1,n},\ g_{m-1,n-1}\}$$

which form a triangle in the Lucas triangle with the peak of the triangle at $g_{m,n}$. Observe that the sum of these three numbers does not depend on their position with respect to n. That is, for a given m and n,

$$g_{m,n} + g_{m-1,n} + g_{m-1,n-1} = g_{m,\ell} + g_{m-1,\ell} + g_{m-1,\ell-1},$$

for all $1 \leqslant \ell,\ n \leqslant m - 1$.

Furthermore, note that the sum of the three numbers forming such triangles for a given m always equals the sum of a Lucas and Fibonacci number associated with the given m. That is,

$$g_{m,n} + g_{m-1,n} + g_{m-1,n-1} = L_{m-1} + F_{m+1}.$$

Finally, we examine the three numbers

$$\{g_{m,n},\ g_{m,n-1},\ g_{m-1,n-1}\}$$

which also form a triangle but with the peak at $g_{m-1,n-1}$. The sum of numbers forming the base minus the peak number is constant with regard to horizontal motion and it is again the sum of a Lucas and Fibonacci number. That is,

$$g_{m,n} + g_{m,n-1} - g_{m-1,n-1} = L_{m-2} + F_m = 4F_{m-2}.$$

3. GRAPHICAL EQUIVALENT OF LUCAS TRIANGLE

In [4], we find a nonadjacent number $P(G, k)$ for a graph defined as the number of ways in which k disconnected lines are chosen from G. Furthermore, in [5], we find the definition of a topological index for nondirected graphs. This is a unique number associated with a given nondirected graph.

The topological index for a linear graph is given in [4], and it is shown to be a Fibonacci number (Table 1). Similarly, in the same manuscript, it is shown that the topological index for a cyclic graph is a Lucas number (Table 3). Using these concepts, Hosoya [2] defines a Fibonacci triangle for the sequence $\{F_{n,m}\}$ and constructs what is called *the graphical equivalent of the Fibonacci triangle* by letting $f_{n,m}$ be the index of a graph and then replacing $f_{n,m}$ with its graph. To do this, he also uses the composition principles defined in [4].

Adopting the procedures of Hosoya in [2], we replace each number $g_{m,n}$ in the Lucas triangle by its graph, obtaining *the graphical equivalent of the Lucas triangle* shown in Figure 2. Note that all the linear and cyclic graphs must occur in Figure 2.

The graphical equivalent of the Lucas triangle easily leads to its corresponding topological index, which can be used in the chemistry of organic substances [5] when dealing with the boiling point to determine the structure of saturated hydrocarbons.

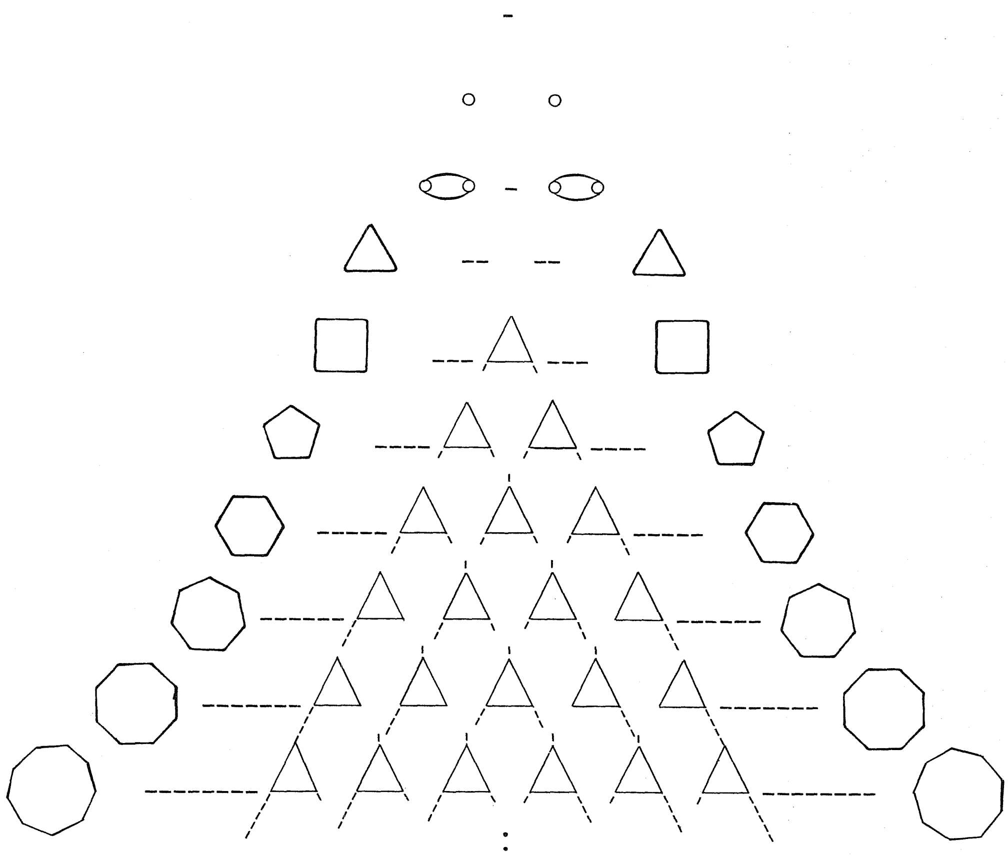

FIGURE 2. GRAPHICAL EQUIVALENT OF LUCAS TRIANGLE

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