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ON FIBONACCI NUMBERS WHICH ARE POWERS: II

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INTRODUCTION

Consider the equation:

$$F_m = c^t \tag{*}$$

where F_m denotes the mth Fibonacci number, and $c^t > 1$. Without loss of generality, we may require that t be prime. The unique solution for $t = 2$, namely $(m, c) = (12, 12)$, was given by J. H. E. Cohn [2], and by O. Wyler [11]. The unique solution for $t = 3$, namely $(m, c) = (6, 2)$, was given by H. London and R. Finkelstein [5] and by J. C. Lagarias and D. P. Weisser [4]. A. Petho [6] showed that (*) has only finitely many solutions with $t > 1$, where m, c, t all *vary*. In fact, he shows that all solutions of (*) can be effectively determined; that is, there is an effectively computable bound B such that all solutions of (*) have

$$\max(|m|, |c|, t) < B. \tag{**}$$

Similar results were obtained independently by C. L. Stewart [10], see, also, T. N. Shorey and C. L. Stewart [9]. The proofs of these results use lower estimates on linear forms in the logarithms of algebraic numbers due to A. Baker [1], and the bounds obtained for B in (**) are astronomical. In [7], A. Petho claims that (*) has no solutions for $t = 5$.

In [8], we showed that if $m = m(t)$ is the least natural number for which (*) holds for given t, then m is odd. In this paper, our main result, which we obtained by elementary methods, is that m must be prime. If (*) has solutions for $t > 5$, and if q is a prime divisor of F_m, one would therefore have $z(q^t) = z(q) = m$, where $z(q)$ denotes the Fibonacci entry point of q. This requirement casts doubt on the existence of such solutions. For the sake of convenience, we occasionally write $F(m)$ instead of F_m.

PRELIMINARIES

(1) If t is a given prime, $t \geqslant 5$, and $m = m(t)$ is the least natural number such that (*) holds, then m is odd.

(2) $F_j \mid F_{jk}$

(3) $(F_j, F_k) = F_{(j,k)}$

(4) $(F_j, F_{jk}/F_j) | k$

(5) $F_1 = 1$

(6) $5^j \| k$ iff $5^j \| F_k$

(7) If p is an odd prime, then $p^2 \nmid F(p^j k)/F(p^{j-1}k)$

(8) If $xy = z^n$, n is odd, and $(x, y) = 1$, then $x = u^n$, $y = v^n$, where $(u, v) = 1$ and $uv = z$.

(9) If $xy = z^n$, n is odd, p is prime, $(x, y) = p$, and $p^2 \nmid y$, then $x = p^{n-1}u^n$, $y = pv^n$, where $(u, v) = (p, v) = 1$.

(10) If $2^k | F_m$, where $k \geqslant 3$, then $3 * 2^{k-2} | m$

(11) If p is prime, then $p | F_{p - e_p}$, where $e_p = \begin{cases} 1, & \text{if } p \equiv \pm 1 \pmod{10}, \\ 0, & \text{if } p = 5, \\ -1, & \text{otherwise.} \end{cases}$

(12) $F_j < F_{jk}$ if $j \geqslant 2$ and $k \geqslant 2$

Remarks: All but (1) and (4) are elementary and/or well known. (1) is the Corollary to Theorem 1 in [8], and (4) is Lemma 16 in [3].

THE MAIN RESULTS

Theorem 1

If t is a given prime, $t \geqslant 5$, and $m = m(t)$ is the least natural number such that $F_m = c^t > 1$, then m is prime.

Proof: Let

$$m = \prod_{i=1}^{r} p_i^{e_i},$$

where the p_i are primes and $p_1 < p_2 < \cdots < p_r$ if $r > 1$. Furthermore, assume m is composite, so that $p_r < m$. (1) implies $2 < p_1$. Let

$$d = (F(p_r), F(m)/F(p_r)).$$

(4) implies $d | (m/p_r)$. If $d = 1$, then since hypothesis implies

$$F(p_r) * F(m)/F(p_r) = c^t,$$

(8) and (12) imply $F(p_r) = a^t$ with $1 < a < c$, contradicting the minimality of m. If $d > 1$, then $p_i | d$ for some i such that $1 \leqslant i \leqslant r$. If $i < r$, then Lemma 1, which is proved below, implies $p_i = 2$, a contradiction. If $i = r$, then (11) implies $p_r = 5$, so $r = 1$ or 2. If $r = 2$, then $m = 3^a 5^b$. But $F_3 = 2$, so the

hypothesis and (2) imply $2|c^t$, hence $2^t|c^t$, and $2^t|F_m$. Now (10) implies that $3*2^{t-2}|3^a5^b$, so that $t = 2$, a contradiction. If $r = 1$, then $m = 5^e$, which is impossible by Lemma 3, which is proved below.

Lemma 1

If p, q are primes such that $p < q$ and $p|F(q^k)$ for some k, then $p = 2$ and $q = 3$.

Proof: The hypothesis, (11), and (3) imply $p|F_d$, where $d = (q^k, p - e_p)$. (5) implies $d > 1$, so that $d = q^j$ for some j such that $1 \leqslant j \leqslant k$. Therefore, $q^j|(p - e_p)$, so that $q \leqslant q^j \leqslant p + 1$. But the hypothesis implies $q \geqslant p + 1$. Therefore, $q = p + 1$, so that $p = 2$ and $q = 3$.

Lemma 2

If $F(5^j) = 5^j v_j^e$, where $5 \nmid v_j$, then $F(5^{j-1}) = 5^{j-1}v_{j-1}^e$, where $5 \nmid v_{j-1}$.

Proof: The hypothesis and (2) imply $F(5^{j-1}) * F(5^j)/F(5^{j-1}) = 5^j v_j^e$. (6) and (7) imply

$$(F(5^{j-1}),\ F(5^j)/F(5^{j-1})) = 5,$$

so that (9) implies $F(5^{j-1}) = 5^{j-1}v_{j-1}^e$, and (6) implies $5 \nmid v_{j-1}$.

Lemma 3

$F(5^j) \neq c^t$ for $t > 1$.

Proof: If $F(5^j) = c^t$, then (6) implies $5^j d = c^t$, where $5 \nmid d$. Now (8) implies $5^j = u^t$, $d = v_j^t$, so that $F(5^j) = 5^j v_j^t$. Applying Lemma 2 $j - 2$ times, one obtains $F(5^2) = 5^2 v_2^t$. But $F(5^2)/5^2 = 3001$, so that $v_2^t = 3001$, a contradiction, since 3001 is prime.

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