◆◇◆◇◆

ADDITIVE PARTITIONS OF THE POSITIVE INTEGERS AND GENERALIZED FIBONACCI REPRESENTATIONS

V. E. HOGGATT, JR.
(Deceased)

MARJORIE BICKNELL-JOHNSON
San Jose State University, San Jose, CA 95192

(Submitted April 1980)

ABSTRACT

The generalized Fibonacci numbers $\{u_n\}$,

$$u_{n+2} = u_{n+1} + u_n,\ u_1 = a,\ u_2 = b,\ (a,\ b) = 1,$$

induce a unique additive partition of the set of positive integers formed by two disjoint subsets such that no two distinct elements of either subset have u_n as their sum. We examine the values of a special function

$$E_n(m) = mu_{n-1} - u\,[mu_{n-1}/u_n],\ m = 1,\ 2,\ \ldots,\ u_n - 1,\ n \geqslant 2,$$

and find relationships to the additive partition of N as well as to Wythoff's pairs and to representations of integers using the double-ended sequence $\{u_n\}_{-\infty}^{\infty}$ and the extended sequence $\{u_n\}_{-m}^{\infty}$. We write a Zeckendorf theorem for double-ended sequences and show completeness for the extended sequences.

1. TABULATION OF $E_n(m)$ FOR THE FIBONACCI AND LUCAS SEQUENCES

We begin with the ordinary Fibonacci sequence $\{F_n\}$, where $F_1 = 1$, and $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$. We tabulate and examine a special function $E_n(m)$, defined by

$$E_n(m) = mF_{n-1} - F_n[mF_{n-1}/F_n],\ m = 1,\ 2,\ \ldots,\ F_n - 1,\ n \geqslant 2, \tag{1.1}$$

where $[x]$ is the greatest integer function. Notice that $n = 2$ gives the trivial $E_2(m) = 0$ for all m, while $E_3(m)$ is 1 for m odd and 0 for m even.

The table of values for $E_n(m)$ (Table 1.1) reveals many immediate patterns. First, $E_n(m)$ is periodic with period F_n, and the rth term in the cycle is $E_n(r) = rF_{n-1} \pmod{F_n}$. We could easily show, using properties of modular arithmetic and of the greatest integer function, that

$$E_n(1) = F_{n-1},\ E_n(2) = F_{n-3},\ E_n(3) = L_{n-2},\ E_n(4) = 2F_{n-3}.$$

Also, counting from the end of a cycle, we have

$$E_n(-1) = F_{n-2},\ E_n(-2) = 2F_{n-2},\ E_n(-3) = F_{n-4},\ E_n(-4) = L_{n-3},$$

which also can be established by elementary methods, but these apparent patterns are not the main thrust of this paper.

TABLE 1.1

VALUES OF $E_n(m)$ FOR THE FIBONACCI SEQUENCE

	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
$E_n(m)$:	$2m - 3\left[\frac{2m}{3}\right]$	$3m - 5\left[\frac{3m}{5}\right]$	$5m - 8\left[\frac{5m}{8}\right]$	$8m - 13\left[\frac{8m}{13}\right]$	$13m - 21\left[\frac{13m}{21}\right]$
$m = 1$	2	3	5	8	13
$m = 2$	1	1	2	3	5
$m = 3$	0	4	7	11	18
$m = 4$	2	2	4	6	10
$m = 5$	1	0	1	1	2
$m = 6$	0	3	6	9	15
$m = 7$	2	1	3	4	7
$m = 8$	1	4	0	12	20
$m = 9$	0	2	5	7	12
$m = 10$	2	0	2	2	4
$m = 11$	1	3	7	10	17
$m = 12$	0	1	4	5	9
$m = 13$	2	4	1	0	1
$m = 14$	1	2	6	8	14
$m = 15$	0	0	3	3	6
$m = 16$	2	3	0	11	19
$m = 17$	1	1	5	6	11
$m = 18$	0	4	2	1	3
$m = 19$	2	2	7	9	16
$m = 20$	1	0	4	4	8
$m = 21$	0	3	1	12	0

We need two other number sequences, derived from the Fibonacci numbers. We write the disjoint sets $\{A_n\}$ and $\{B_n\}$, which are formed by making a partition of the positive integers such that no two distinct members from A and no two distinct members from B have a sum which is a Fibonacci number. We also write the first few Wythoff pairs (a_n, b_n) [1] for inspection.

n	A_n	B_n
1	1	2
2	3	4
3	6	5
4	8	7
5	9	10
6	11	12
7	14	13
8	16	15
9	17	18
10	19	20
...	...	...

n	a_n	b_n
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23
10	16	26
...	...	...

We note that the Wythoff pairs are given by

$$a_n = [n\alpha] \quad \text{and} \quad b_n = [n\alpha^2], \tag{1.2}$$

where $[x]$ is the greatest integer contained in x and $\alpha = (1+\sqrt{5})/2$ is the Golden Section ratio. Also, $b_n = a_n + n$, and a_n is the smallest integer not yet used. It is also true that no two distinct b_n's have a Fibonacci number as their sum, and that $\{b_n\} \subset \{B_n\}$.

Now, examine the periods of the values of $E_n(m)$:

$n = 5$:

$$\left.\begin{matrix}3\\1\\4\end{matrix}\right\} a_n \qquad \left.\begin{matrix}3\\1\end{matrix}\right\} A_n$$

$$\left.\begin{matrix}2\end{matrix}\right\} b_n \qquad \left.\begin{matrix}4\\2\end{matrix}\right\} B_n$$

$n = 6$:

$$\left.\begin{matrix}5\\2\\7\end{matrix}\right\} b_n \qquad \left.\begin{matrix}5\\2\\7\\4\end{matrix}\right\} B_n$$

$$\left.\begin{matrix}4\\1\\6\\3\end{matrix}\right\} a_n \qquad \left.\begin{matrix}1\\6\\3\end{matrix}\right\} A_n$$

$n = 7$:

$$\left.\begin{matrix}8\\3\\11\\6\\1\\9\\4\\12\end{matrix}\right\} a_n \qquad \left.\begin{matrix}8\\3\\11\\6\\1\\9\end{matrix}\right\} A_n$$

$$\left.\begin{matrix}7\\2\\10\\5\end{matrix}\right\} b_n \qquad \left.\begin{matrix}4\\12\\7\\2\\10\\5\end{matrix}\right\} B_n$$

$n = 8$:

$$\left.\begin{array}{r}13\\5\\18\\10\\2\\15\\7\\20\end{array}\right\} b_n \qquad \left.\begin{array}{r}13\\5\\18\\10\\2\\15\\7\\20\\12\\4\end{array}\right\} B_n \qquad \left.\begin{array}{r}12\\4\\17\\9\\1\\14\\6\\19\\11\\3\\16\\8\end{array}\right\} a_n \qquad \left.\begin{array}{r}17\\9\\1\\14\\6\\19\\11\\3\\16\\8\end{array}\right\} A_n$$

Notice that the integers 1, 2, 3, ..., $F_n - 1$, all appear, but not in natural order. Each cycle is made up of early values of $\{a_n\}$ and $\{b_n\}$, and of early values of $\{A_n\}$ and $\{B_n\}$, not in order, but without omissions.

If we apply $E_n(m)$ to the Lucas numbers L_n, defined by $L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_{n+1} + L_n$, so that we consider

$$E_n(m) = mL_{n-1} - L_n[mL_{n-1}/L_n],\ m = 1, 2, \ldots, L_n - 1, \tag{1.3}$$

then we get the integers 1, 2, 3, ..., $L_n - 1$ in some order. Recall our generalized Wythoff numbers a_n, b_n, and c_n [1, p. 200]. We obtain within each cycle a segment of $\{a_n\}$, a segment of $\{c_n\}$, and a segment of $\{b_n\}$, where each segment is complete (the first few terms of each sequence without omission, but not in order). This same cycle contains the first few terms of $\{A_n\}$, out of order, but without omissions, followed by the first few terms of $\{B_n\}$, where $\{A_n\}$ and $\{B_n\}$ is the unique split of the positive integers induced by the Lucas sequence such that no two elements of $\{A_n\}$, and no two elements of $\{B_n\}$, have a Lucas number for their sum.

To illustrate the Lucas case, we write the first twelve values of the generalized Wythoff numbers, and early values of the partition sets:

n	a_n	b_n	c_n
1	1	3	2
2	4	7	6
3	5	10	9
4	8	14	13
5	11	18	17
6	12	21	20

n	A_n	B_n
1	1	2
2	4	3
3	5	6
4	8	7
5	9	10
6	11	13

n	a_n	b_n	c_n
7	15	25	24
8	16	28	27
9	19	32	31
10	22	36	35
11	23	39	38
12	26	43	42

n	A_n	B_n
7	12	14
8	15	17
9	16	18
10	19	20
11	22	21
12	23	24
13	26	25
14	27	28

Now, examine the periods of values of $E_n(m)$ for the Lucas sequence:

$n = 4$: $4m - 7[4m/7]$

$$\left.\begin{matrix}4\\1\\5\end{matrix}\right\} a_n \quad \left.\begin{matrix}2\\6\end{matrix}\right\} c_n \quad \left.\begin{matrix}3\end{matrix}\right\} b_n \qquad\qquad \left.\begin{matrix}4\\1\\5\end{matrix}\right\} A_n \quad \left.\begin{matrix}2\\6\\3\end{matrix}\right\} B$$

$n = 5$: $7m - 11[7m/11]$

$$\left.\begin{matrix}7\\3\\10\end{matrix}\right\} b_n \quad \left.\begin{matrix}6\\2\\9\end{matrix}\right\} c_n \quad \left.\begin{matrix}5\\1\\8\\4\end{matrix}\right\} a_n \qquad\qquad \left.\begin{matrix}7\\3\\10\\6\\2\end{matrix}\right\} B_n \quad \left.\begin{matrix}9\\5\\1\\8\\4\end{matrix}\right\} A_n$$

$n = 6$: $11m - 18[11m/18]$

$$\left.\begin{matrix}11\\4\\15\\8\\1\\12\\5\\16\end{matrix}\right\} a_n \quad \left.\begin{matrix}9\\2\\13\\6\\17\end{matrix}\right\} c_n \quad \left.\begin{matrix}10\\3\\14\\7\end{matrix}\right\} b_n \qquad\qquad \left.\begin{matrix}11\\4\\15\\8\\1\\12\\5\\16\\9\end{matrix}\right\} A_n \quad \left.\begin{matrix}2\\13\\6\\17\\10\\3\\14\\7\end{matrix}\right\} B_n$$

$n = 7$: $18m - 29[18m/29]$

$$\left.\begin{matrix}18\\7\\25\\14\\3\\21\\10\\28\end{matrix}\right\} b_n \quad \left.\begin{matrix}17\\6\\24\\13\\2\\20\\9\\27\end{matrix}\right\} c_n \quad \left.\begin{matrix}16\\5\\23\\12\\1\\19\\8\\26\\15\\4\\22\end{matrix}\right\} a_n \qquad\qquad \left.\begin{matrix}18\\7\\25\\14\\3\\21\\10\\28\\17\\6\\24\\13\\2\\20\end{matrix}\right\} B_n \quad \left.\begin{matrix}9\\27\\16\\5\\23\\12\\1\\19\\8\\26\\15\\4\\22\end{matrix}\right\} A_n$$

For comparison, the generalized Wythoff numbers are formed by letting a_n be the smallest positive integer not yet used, letting $c_n = b_n - 1$, and forming $b_n = a_n + d_n$, where $d_n \neq b_k + 1$. Letting the generalized Wythoff numbers be denoted with an asterisk, we can express them in terms of Wythoff pair numbers as

$$a_n^* = 2a_n - n, \quad b_n^* = b_n + n = a_n + 2n, \quad c_n^* = a_n + 2n - 1 = a_{a_n} + n.$$

It is also true that $a_i^* + a_j^* \neq L_m$, $b_i^* + b_j^* \neq L_m$, and the Lucas generalized Wythoff numbers and the Lucas partition sets have the subset relationships $\{a_n\} \subset \{A_n\}$ and $\{b_n\} \subset \{B_n\}$.

2. ZECKENDORF THEOREM FOR DOUBLE-ENDED SEQUENCES

Before considering representations and additive partitions regarding the generalized Fibonacci sequence $\{u_n\}_{-m}^{\infty}$, where $u_1 = a$ and $u_2 = b$, $u_{n+2} = u_{n+1} + u_n$, we consider representations of the integers in terms of specialized $\{u_n\}$, where $u_1 = 1$ and $u_2 = p$.

Theorem 2.1 (Zeckendorf Theorem for double-ended sequences): Let $p \geqslant 1$ be a positive integer, and let $u_{n+2} = u_{n+1} + u_n$, $u_1 = 1$, $u_2 = p$. Then every positive integer has a representation from $\{u_n\}_{-\infty}^{\infty}$, provided that no two consecutive u_j are in the same representation.

Proof: We need to recall two major results from earlier work. David Klarner [2] has proved

Klarner's Theorem: Given the nonnegative integers A and B, there exists a unique set of integers $\{k_1, k_2, k_3, \ldots, k_r\}$ such that

$$A = F_{k_1} + F_{k_2} + \cdots + F_{k_r}$$
$$B = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1}$$

for $|k_i - k_j| \geqslant 2$, $i \neq j$, where each F_i is an element of the sequence $\{F_i\}_{-\infty}^{\infty}$.

When $u_1 = 1$ and $u_2 = p$, we know from earlier work that

$$u_{n+1} = pF_n + F_{n-1},$$

for all integral n. Next, if we wish a representation of an integer $m > 0$, we merely solve

$$A = 0 = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1}$$

$$B = m = F_{k_1} + F_{k_2} + \cdots + F_{k_r}$$

which has a unique solution by Klarner's Theorem. A constructive method of solution is given in [3]. Thus,

$$\begin{aligned} m &= u_{k_1+1} + u_{k_2+1} + \cdots + u_{k_r+1} \\ &= p(F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1}) + (F_{k_1} + F_{k_2} + \cdots + F_{k_r}) \end{aligned}$$

We note in passing that the representation we now have is independent of the explicit $p > 0$.

Theorem 2.2: The Fibonacci extended sequence is complete with respect to the integers.

Proof: Since 1, 2, 3, 5, 8, 13, ..., is complete with respect to the positive integers, one notes

$$F_{-n} = (-1)^{n+1}F_n,$$

and, therefore, one can pick out an arbitrarily large negative Fibonacci number. Consider M an arbitrary negative integer, and there exists a Fibonacci number F_{-k} such that $F_{-k} < M < 0$. Now, $M - F_{-k} = N$, which is positive and has a Zeckendorf representation using Fibonacci numbers, and $M = N + F_{-k}$ is the representation we seek.

Since $u_{n+2} = u_{n+1} + u_n$, if it consists of positive integers as $n \to \infty$, then, as $n \to -\infty$, the terms become alternating and negatively very large. Thus, the same thing holds for the generalized Fibonacci numbers once we know that they are complete with respect to the positive integers, finishing the proof of Theorem 2.1.

Completeness of the generalized sequence $\{u_n\}_{-\infty}^{\infty}$ is equivalent to showing that every positive integer is expressible as the sum of a subsequence $\{u_n\}_{-m}^{\infty}$, $m > 0$, where m is independent of the integer chosen. We show some special cases:

Case 1: $p = 1$ 1, 1, 2, 3, 5, ... Already complete.
Case 2: $p = 2$ 1, 2, 3, 5, 8, ... Already complete.
Case 3: $p = 3$ 1, 3, 4, 7, 11, ...
Complete when $L_0 = 2$ is added to the sequence.

Case 4: $p = 4$ — 1, 4, 5, 9, 14, 23, ... Complete when $u_0 = 3$ and $u_{-1} = -2$ are added to the sequence.

Case 5: $p = 5$ — 1, 5, 6, 11, 17, ... Complete when $u_0 = 4$ and $u_{-1} = -3$ are added to the sequence.

Case 6: $p = 6$ — 1, 6, 7, 13, 20, ... Complete when $u_0 = 5$, $u_{-1} = -4$, and $u_{-2} = 9$ are added.

Case 7: $p = 7$ — 1, 7, 8, 15, 23, ... Complete when $u_0 = 6$, $u_{-1} = -5$, $u_{-2} = 11$, and $u_{-3} = -16$ are added.

Case 8: $p = 8$ — 1, 8, 9, 17, 26, ... Becomes complete when $u_0 = 7$, $u_{-1} = -6$, $u_{-2} = 13$, and $u_{-3} = -19$ are added.

Next we consider the generalized Fibonacci sequence.

Theorem 2.3: Let $u_{n+2} = u_{n+1} + u_n$, where $u_1 = a$, $u_2 = b$, and $(a, b) = 1$, $b \geqslant a \geqslant 1$. Then, every positive integer has a representation from $\{u_n\}_{-\infty}^{\infty}$ provided that no two consecutive u_j are in the same representation.

Proof: It is known that the generalized Fibonacci numbers are related to the ordinary Fibonacci numbers by

$$u_{n+1} = bF_n + aF_{n-1}. \tag{2.1}$$

Let m be a positive integer, $m \geqslant b$. Then we can always write

$$m = bA + aB$$

for some integers A and B, since $(a, b) = 1$. If both A and B are nonnegative, we are done, since the dual representation of A and B, by Klarner's Theorem, leads to a representation of m via (2.1). If A or B is negative, notice that, since the ordered pair (A, B) is a lattice point for a line with slope $-b/a$ and y-intercept m/a, if we can add an arbitrarily large integer to m, then we can raise the line so that it crosses the first quadrant and we will have nonnegative values for A and B. Thus, choose $u_{-k} < 0$ with an absolute value sufficiently large, and we represent

$$m - u_{-k} = bA^* + aB^*$$

for A^* and B^* nonnegative. We then represent $m - u_{-k}$ via Klarner's Theorem, and add u_{-k} to that representation to represent m. Similarly, if $m < b$, since the negatively subscripted terms of u_n become negatively as large as we please, choose $u_{-k} < 0$ so that $m - u_{-k} > b$, represent $m - u_{-k}$

as above, and then add u_{-k} to the representation.

3. A PATTERN ARISING FROM KLARNER'S DUAL ZECKENDORF REPRESENTATION

Recall the Klarner dual Zeckendorf representation given in Section 2, where

$$A = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1} = 0$$

$$B = F_{k_1} + F_{k_2} + \cdots + F_{k_r} = n,$$

where $n = 1, 2, 3, \ldots,$ $|k_i - k_j| \geqslant 2,$ $i \neq j,$ and F_j comes from $\{F_j\}_{-\infty}^{\infty}$. The constructive method for solving for the subscripts k_j to represent A and B described in our earlier work [3] leads to a symbolic display with a generous sprinkling of Lucas numbers. Here we use only two basic formulas,

$$u_{n+2} = u_{n+1} + u_n \quad \text{and} \quad 2u_n = u_{n+1} + u_{n-2}.$$

This allows us to push both right and left. We continue to add $F_{-1} = 1$ at each step, using the rules given to simplify the result. For example, for $n = 1$, we have $F_{-1} = 1$. For $n = 2$, $F_{-1} + F_{-1} = 2F_{-1} = F_0 + F_{-3} = 2$. For $n = 3$, $F_{-1} + F_0 + F_{-3}$ becomes $F_1 + F_{-3} = 1 + 2 = 3$. We display Table 3.1.

Strangely enough, the Wythoff pairs sequences enter into this again. The basic column centers under F_{-1}. The display is for expressions for B only; A is a translation of one space to the right. At each step, $B = n$, and $A = 0$.

From Table 3.1, many patterns are discernible. There are always the same number of successive entries in a given column. Under F_{-2} there are L_1; under F_{-3}, L_2; under F_{-4}, L_3; and under F_{-5}, L_4. Under F_{-6} there are L_5 successive entries, starting with $B = 30$, and under F_{-7} there are L_6 successive entries. On the line for $B = 47$, there are only two entries, one corresponding to $F_{-9} = 34$ and one to $F_7 = 13$, so that $34 + 13 = 47$ as required, while $F_{-8} = -21$ and $F_8 = 21$ have a zero sum as required.

The columns to the right of F_{-1} (under F_0, for instance) have $L_n \pm 1$ alternately successive entries, but the same numbers of successive entries always appear in the columns. Once we have all spaces cleared except the extreme edges in the pattern being built, we start again in the middle, as in line 48 or line 19.

TABLE 3.1

B	F_{-9}	F_{-8}	F_{-7}	F_{-6}	F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6
1									x							
2							x			x						
3							x				x					
4							x		x		x					
5					x			x				x				
6					x					x		x				
7					x								x			
8					x				x				x			
9					x		x			x			x			
10					x		x				x		x			
11					x		x		x		x		x			
12			x			x		x						x		
13			x			x				x				x		
14			x			x					x			x		
15			x			x			x		x			x		
16			x					x				x		x		
17			x							x		x		x		
18			x												x	
19			x						x						x	
20			x				x			x					x	
21			x				x				x				x	
22			x				x		x		x				x	
23			x		x			x				x			x	
24			x		x					x		x			x	
25			x		x								x		x	
26			x		x				x				x		x	
27			x		x		x			x			x		x	
28			x		x		x				x		x		x	
29			x		x		x		x		x		x		x	
30	x			x		x		x								x

4. REPRESENTATIONS AND ADDITIVE PARTITIONS FOR THE SEQUENCE 1, 4, 5, 9, 14, 23, ...

We make the following array:

$$\begin{aligned} A_n &= \text{first positive integer not yet used} \\ B_1(n) &= B_n - 2 \\ B_2(n) &= B_n - 1 \\ B_n &= A_n + d_n, \end{aligned}$$

where $d_n \neq A_j$ and goes through the complement of $\{A_n\}$ in order, except we do not use $B_1(n)$ opposite the second of a consecutive pair of A_n; i.e., we do not use $B_1(n)$ if $A_n = A_{n-1} + 1$. The underlined numbers in the following table cannot be used for d_n.

n	A_n	$B_1(n)$	$B_2(n)$	B_n	d_n
0	0				
1	$\underline{1}$	$\underline{2}$	3	4	3
2	$\underline{5}$	7	8	9	4
3	$\underline{6}$	$\underline{11}$	12	13	7
4	$\underline{10}$	16	17	18	8
5	$\underline{14}$	21	22	23	9
6	$\underline{15}$	$\underline{25}$	26	27	12
7	$\underline{19}$	30	31	32	13
8	$\underline{20}$	$\underline{34}$	35	36	16
9	$\underline{24}$	39	40	41	17
10	$\underline{28}$	44	45	46	18
...	...	...	...	...	...

We now have the following constant differences, where (a_n, b_n) is a Wythoff pair:

$$B_{n+1} - B_n = \begin{cases} 5, & n = a_i \\ 4, & n = b_j \end{cases} \tag{4.1}$$

$$A_{n+1} - A_n = \begin{cases} 4, & n = a_i \\ 1, & n = b_j \end{cases} \tag{4.2}$$

Alternately,

$$A_n = 3a_n - 2n = (2n - a_n) \cdot 1 + (a_n - n) \cdot 4 \tag{4.3}$$

$$B_n = a_n + 3n = (2n - a_n) \cdot 4 + (a_n - n) \cdot 5 \tag{4.4}$$

Apparently, $d_n \neq B_j + 1$ and $d_n \neq B_j + 2$.

This extends for the sequence $1, p, p+1, \ldots, u_{n+2} = u_{n+1} + u_n$.

5. REPRESENTATIONS AND ADDITIVE PARTITIONS USING THE GENERALIZED FIBONACCI NUMBERS

We consider the general case for $(a, b) = 1$, and

$$u_1 = a,\ u_2 = b,\ u_{n+2} = u_{n+1} + u_n.$$

First we have a unique additive partition and the function

$$E_n(m) = mu_{n-1} - u_n[mu_{n-1}/u_n], \quad m = 1, 2, \ldots, u_n - 1 \tag{5.1}$$

generates 1, 2, 3, ..., $u_n - 1$, but, of course, not in natural order. One set of the additive partition includes $1 \leqslant m \leqslant u_n/2$, while the other has $u_n/2 < m \leqslant u_n - 1$. Suppose $0 < a^* < b^*$; then the values of $E_n(m)$ are split into b^* disjoint sets whose first elements are 1, 2, 3, ..., a^*, ..., b^*. The elements of the sets to the left of a^* are, correspondingly, 1 less, 2 less, 3 less, ..., as we go to the left, while the sets between a^* and b^* have their values 1 less, 2 less, 3 less, ..., than b^*. Each element satisfies

$$\begin{aligned} a^*_{n+1} - a^*_n &= b, \; n \in \{a_k\}, \; a_k = [k\alpha], \; \alpha = \frac{1 + \sqrt{5}}{2}, \\ a^*_{n+1} - a^*_n &= a, \; n \in \{b_k\}, \; b_k = [k\alpha^2]. \end{aligned} \tag{5.2}$$

The a^*_n are the representations using

$$a + \alpha_2 b + \alpha_3 u_3 + \ldots, \quad \alpha_i = 0 \text{ or } 1,$$

while we can show

$$a^*_n = (2n - a_n)a + (a_n - n)b \tag{5.3}$$

$$b^*_n = (2n - a_n)b + (a_n - n)(a + b) \tag{5.4}$$

because of the formula

$$u_{n+1} = bF_n + aF_{n-1}. \tag{5.5}$$

Let $a^* = \{a + \alpha_2 u_2 + \alpha_3 u_3 + \cdots\}$ in natural order. Then

$$\begin{aligned} a^*_n &= bF_0 + aF_{-1} + \alpha_2(bF_1 + aF_0) + \cdots \\ &= b(F_0 + \alpha_2 F_1 + \alpha_3 F_2 + \cdots) + a(F_{-1} + \alpha_2 F_0 + \cdots) \\ &= a(2n - a_n) + b(a_n - n) \end{aligned}$$

since

$$a_n \rightarrow n \rightarrow a_n - n \rightarrow 2n - a_n.$$

Thus, once we know that $a_{n+1} - a_n = 2$ for $n = a_j$ and $a_{n+1} - a_n = 1$ for $n \neq a_j$, and $u_{n+1} = bF_n + aF_{n-1}$, we have (5.3) and (5.4). Further, these a^*_n and b^*_n are the generalizations of the Wythoff pair numbers $a_n = [n\alpha]$ and $b_n = [n\alpha^2]$ themselves $\left(\alpha = \dfrac{1 + \sqrt{5}}{2}\right)$.

6. REPRESENTATIONS AND ADDITIVE PARTITIONS ARISING FROM TWO SUCCESSIVE FIBONACCI NUMBERS

It is well known that if we start with 1 and 2, we get the Wythoff pairs and have a unique additive partition of the positive integers. Next, to see something else, take 2 and 3. Since (2, 3) = 1, we still have the same additive partition of the positive integers, and the function $E_n(m)$ of (5.1) and (1.1) still yields the residues mod F_n, but our array changes in an interesting way.

TABLE 6.1

2, 3, 5, 8, 13, 21, ...

			A_n	B_n
1	2	3	1	2
4	5	8	3	4
6	7	11	6	5
9	10	16	8	7
12	13	21	9	10
14	15		11	12
...	...	...	14	13
			16	15
a_{a_n}	b_n	a_{b_n}	17	18
$W = 2$		$W = 3$	19	20

Note that, if we give a weight one and b weight two, we have the weights abbreviated by W above. The successive values of $E_n(m)$ are the same as before, but we now have a different split to look at. Note that A and B are formed so that no two elements of either set have 2, 3, 5, 8, 13, ..., as their sum. Now let us look at $E_n(m)$ for $13 = F_{n-1}$ and $21 = F_n$; i.e.,

$$E_n(m) = 13m - 21[13m/21].$$

The twenty values in the cycle are:

13, 5, 18, 10, 2, 15, 7, 20, 12, 4, 17, 9, 1, 14, 6, 19, 11, 3, 16, 8.

The first 10 are elements of B_n; the other 10 are A_n. The first 8 have the form b_n; the next 8 have the form a_{a_n}; the last 4 have the form a_{b_n}.

Now look at the array induced by 3 and 5 as a starting pair.

TABLE 6.2

3, 5, 8, 13, 21, 34, ...

					A_n	B_n
1	2	3	4	5	1	2
6	7	8	12	13	3	4
9	10	11	17	18	6	5
14	15	16	25	26	8	7
19	20	21	33	34	9	10
22	23	24	38	39	11	12
27	28	29	46	47	14	13
			51	52	16	15
...	...	...	...	...	17	18
					19	20
$a_{a_{a_n}}$	b_{a_n}	a_{b_n}	$a_{a_{b_n}}$	b_{b_n}	...	...
$W = 3$			$W = 4$			

The A_n and B_n are the same as before.

Return to the values of $E_n(m)$ for 13 and 21 given above. Notice that the first 3 values—13, 5, 18—come from b_{b_n}; the next 5 from b_{a_n}; the next 3 from $a_{a_{b_n}}$; then five from $a_{a_{a_n}}$; and, lastly, 4 from a_{b_n}. We begin to see familiar patterns emerging [4], [5].

We write the array induced by 5 and 8 in Table 6.3.

TABLE 6.3

5, 8, 13, 21, 34, 55, ...

								A_n	B_n
1	2	3	4	5	6	7	8	1	2
9	10	11	12	13	19	20	21	3	4
14	15	16	17	18	27	28	29	6	5
22	23	24	25	26	40	41	42	8	7
30	31	32	33	34	53	54	55	9	10
35	36	37	38	39	61	62	63	11	12
43	44	45	46	47	74	75	76	14	13
...	...	...	...	...	..	...	...	16	15
$a_{a_{a_a}}$	$b_{a_{a_n}}$	$a_{b_{a_n}}$	$a_{a_{b_n}}$	b_{b_n}	$a_{a_{a_{b_n}}}$	$b_{a_{b_n}}$	$a_{b_{b_n}}$	17	18
Five of weight 4					Three of weight 5			19	20
								...	...

Since a_n and b_n are elements in complementary sets, the array on the left covers the positive integers. Note that the additive partition sequence A_n and B_n is the same for (1, 2), (2, 3), (3, 5), (5, 8), and for all consecutive Fibonacci pairs.

Now, the weights mentioned under the arrays from (2, 3), (3, 5), and (8, 13) are precisely the unshortened sequence of 1's and 2's in the compositions of W (the weight) as laid out by our scheme in [4]. As each must end in a $1 = a_1$ or a $2 = b_1$, to get the proper representation, we simply replace 1 in each case by n and let $n = 1, 2, 3, \ldots$. This is a wonderful application of Wythoff pairs, Fibonacci representations, additive partitions of the positive integers, and the function $E_n(m)$.

Before we prove all of this, we need some results for Wythoff's pairs from [1] and [5]. For Wythoff's pairs (a_n, b_n),

$$a_{b_n+1} - a_{b_n} = 1 \quad \text{and} \quad a_{a_n+1} - a_{a_n} = 2; \tag{6.1}$$

$$a_{a_n} + 1 = b_n. \tag{6.2}$$

Return to the weights given in the array induced by 3 and 5 in Table 6.2. From (6.2), replacing n by a_n, we get immediately that

$$a_{a_{a_n}} + 1 = b_{a_n}. \tag{6.3}$$

Now,

$$a_{a_{a_n}} + 1 = b_{a_n} \quad \text{and} \quad a_{a_{a_n}+1} = a_{a_{a_n}} + 2$$

from (6.1) rewritten as

$$a_{a_n+1} = a_{a_n} + 2.$$

Thus,

$$b_{a_n} + 1 = a_{a_{a_n}+1} = a_{b_n}$$

as required.

We obtain

$$a_{a_{b_n}} + 1 = b_{b_n} \tag{6.4}$$

by replacing n by b_n in (6.2). These are all the weights appearing in Table 6.2.

Now, we move to Table 6.3, the array induced by 5 and 8, and examine the weights. From (6.2), by replacing n by a_{a_n}, we easily obtain

$$a_{a_{a_{a_n}}} + 1 = b_{a_{a_n}}.$$

Then,

$$a_{a_{a_n}} + 2 = b_{a_n} + 1 = a_{a_{a_n}} + 1 = a_{b_{a_n}}.$$

Next, again from (6.2) with n replaced by b_n and then n replaced by a_{b_n}, we have

$$a_{a_{b_n}} + 1 = b_{b_n} \quad \text{and} \quad a_{a_{a_{b_n}}} + 1 = b_{a_{b_n}}.$$

Again using (6.2), we can also write

$$b_{a_{b_n}} + 1 = a_{a_{a_{b_n}}+1} = a_{b_{b_n}}.$$

This undoubtedly continues.

To get our next line of weighted 1's, we simply add a_n to end each 1 of weight 4, and take those of weight 5 together with these. All of the following are of weight 5:

$$a_{a_{a_{a_n}}} \quad b_{a_{a_n}} \quad a_{b_{a_n}} \quad a_{a_{b_{a_n}}} \quad b_{b_{a_n}} \quad a_{a_{a_{b_n}}} \quad b_{a_{b_n}} \quad a_{b_{b_n}}$$

The five on the left end in a_n, and came from adding an a_n to each 1 of weight 4; the three on the left ending in b_n are of weight 5 already.

To get the next five of weight 6 to the right, we add a_{b_n} to the end of the weight 3 a_{a_n}, b_{a_n}, and a_{b_n}, then add b_n only to the weight 4 $a_{a_{b_n}}$ and b_{b_n} of Table 6.2, to form

$$a_{a_{a_{b_n}}} \quad b_{a_{a_{b_n}}} \quad a_{b_{a_{b_n}}} \quad \text{and} \quad a_{a_{b_{b_n}}} \quad b_{b_{b_n}}.$$

Now, we would like to have

$$b_{b_{a_n}} + 1 = a_{a_{a_{b_n}}}. \tag{6.5}$$

From $a_{a_{b_{a_n}}} + 1 = b_{b_{a_n}}$, we have

$$a_{a_{b_{a_n}}} + 2 = b_{b_{a_n}} + 1 = a_{a_{b_{a_n}}+1}.$$

Now, $a_{b_{a_n}} + 1 = a_{b_{a_n}+1}$, so that

$$a_{b_{a_n}} + 1 = a_{b_{a_n}+1} = a_{a_{a_n}+2} = a_{a_{a_n}+1} = a_{a_{b_n}}.$$

Thus, $a_{a_{b_{a_n}}+1} = a_{a_{a_{b_n}}}$, establishing (6.5).

Finally, we write a complete proof based on (6.1) and (6.2). Notice that we have to show that the differences between successive columns are always 1, except for the transition that comes between the columns headed by F_{n-1} and $F_{n-1} + 1$ in the array. Also, we need a rule for formuation.

The rule of formation of one array from the preceding array is as follows: To get the array with F_{n+1} columns, build up the left F_{n-1} and the right F_{n-2} columns of the array for F_n columns by extending the subscripts. Add a_n to the bottom of each subscript in the left part, copy down the right part next as is, and then copy down the old left part with b_n added to the bottom to get the rew right part.

					Left Part	|	Right Part		
Line 1					a_n	|	b_n		
Line 2				a_{a_n}	b_n	|	a_{b_n}		
Line 3			$a_{a_{a_n}}$	b_{a_n}	a_{b_n}	|	$a_{a_{b_n}}$	b_{b_n}	
Line 4	$a_{a_{a_{a_n}}}$	$b_{a_{a_n}}$	$a_{b_{a_n}}$	$a_{a_{b_n}}$	b_{b_n}	|	$a_{a_{a_{b_n}}}$	$b_{a_{b_n}}$	$a_{b_{b_n}}$

From [4, p. 315], $$F_{2n+2} = \underbrace{a_{b_{b_{b_{\cdot_{\cdot_{\cdot_{b_1}}}}}}}}_{n\ b\text{'s}} \quad \text{and} \quad F_{2n+1} = \underbrace{b_{b_{b_{\cdot_{\cdot_{\cdot_{b_1}}}}}}}_{n\ b\text{'s}}$$

Now, the entries before the dashed line in Lines 1, 2, 3, and 4 above are alternately odd- and even-subscripted Fibonacci numbers, while the entries on the far right are the next higher Fibonacci numbers if we replace n by 1. Thus, we have the sequence of representations in natural order.

We show that the columns always differ by one within the left part and within the right part. We count each a subscript 1 and each b subscript 2. Then the left part of Line 1 has weight 1 and the right part weight 2, and the left part of Line 2 has weight 2 and the right part has weight 3. The columns in the left part of Line 2 differ by 1 according to (6.2),

$$a_{a_n} + 1 = b_n,$$

which generalizes to

$$a_{a_{A_n}} + 1 = b_{A_n} \tag{6.6}$$

for any A_n. Next, for Line 3,

$$\begin{array}{ccc|cc} a_{a_{a_n}} & b_{a_n} & a_{b_n} & a_{a_{b_n}} & b_{b_n} \end{array}$$

Weight 3 Weight 4

we have, from (6.2),

$$b_{a_n} + 1 = a_{a_{a_n}} + 2 = a_{a_{a_n}+1} = a_{b_n},$$

and from (6.1) and (6.6),

$$b_{a_{A_n}} + 1 = a_{a_{a_{A_n}}} + 2 = a_{a_{a_{A_n}}+1} = a_{b_{A_n}}, \tag{6.7}$$

so that

$$a_{a_{b_n}} + 1 = b_{b_n},$$

follows for $A_n = b_n$.

Now, for

$$\begin{array}{ccccc|ccc} a_{a_{a_{a_n}}} & b_{a_{a_n}} & a^*_{b_{a_n}} & a_{a_{b_n}} & b_{b_n} & a_{a_{a_{b_n}}} & b_{a_{b_n}} & a_{b_{b_n}} \end{array}$$

Weight 4 Weight 5

note that all cases follow from earlier cases, except transition case #1, marked with an asterisk above:

$$a_{b_{a_n}} + 1 = a_{a_{b_n}}. \tag{6.8}$$

But, $a_{b_{a_n}} + 1 = a_{b_{a_n}+1} = a_{a_{a_n}+2} = a_{a_{a_n+1}} = a_{a_{b_n}}$.

We now display all of one more case:

$$\begin{array}{cccccccc|ccccc} a_{a_{a_{a_{a_n}}}} & b_{a_{a_{a_n}}} & a_{b_{a_{a_n}}} & a_{a_{b_{a_n}}} & b^*_{b_{a_n}} & a_{a_{a_{b_n}}} & b_{a_{b_n}} & a_{b_{b_n}} & a_{a_{a_{a_{b_n}}}} & b_{a_{a_{b_n}}} & a_{b_{a_{b_n}}} & a_{a_{b_{b_n}}} & b_{b_{b_n}} \end{array}$$

Weight 5 Weight 6

Note that all these columns differ by one within the left and right parts from earlier results, except transition case #2, marked with an asterisk above,

$$b_{b_{a_n}} + 1 = a_{a_{a_{b_n}}}, \tag{6.9}$$

which is proved as follows:

$$b_{b_{a_n}} + 1 = a_{a_{b_{a_n}}} + 1 + 1 = a_{a_{b_{a_n}}} + 2 = a_{a_{b_{a_n}}+1} = a_{a_{a_{b_n}}}$$

from (6.2) and (6.8), which was transition case #1 above.

When we write the next line, our transition case will again be like #1, as

$$a_{b_{b_{a_n}}} + 1 = a_{a_{a_{a_{b_n}}}}, \tag{6.10}$$

proved from (6.1) and (6.9), which was transition case #2 above, as

$$a_{b_{b_{a_n}}} + 1 = a_{b_{b_{a_n}}} = a_{a_{a_{b_n}}}.$$

Next, the transition will again by like #2 above,

$$b_{b_{b_{a_n}}} + 1 = a_{a_{a_{a_{a_{b_n}}}}}, \tag{6.11}$$

proved from transition case #1 given in (6.10):

$$b_{b_{b_{a_n}}} + 1 = a_{a_{b_{b_{a_n}}}} + 2 = a_{a_{b_{b_{a_n}}}+1} = a_{a_{a_{a_{a_{b_n}}}}}.$$

The proof is now complete, by the principle of mathematical induction. That is, if, for the earlier cases, each term of the sequence plus 1 is the next one to the right, then, from the formation rules and general result (6.1), we get that it holds for the next case, but we have to prove the transition cases, since we get the results for each left and right part separately. To get a new left section, we add a_n to the bottom of the old left section subscripts and use general result (6.7) and just copy down the right section as is, and these two parts are, separately, okay. The transition from left to right in the new left section is now proved in four cases by mathematical induction. The new right part, which is the former left part with b_n added on the end, yields to general result (6.1). This completes the discussion.

Suppose the line array, at some level, produces sequences whose elements cover the positive integers without overlap. Since a_n is added to the left portion to form the left part of the new left part and the b_n is added to the left portion to form the new right part, these two pieces cover all the integers together that we covered by the left part before, and the old right side is left intact, ao that the new line again covers the positive integers without overlap.

REFERENCES

1. V. E. Hoggatt, Jr., Marjorie Bicknell-Johnson, & Richard Sarsfield. "A Generalization of Wythoff's Game." *The Fibonacci Quarterly* 17, No. 3 (Oct. 1979):198-211.
2. David A. Klarner. "Partitions of N into Distinct Fibonacci Numbers." *The Fibonacci Quarterly* 6, No. 4 (Oct. 1968):235-43.
3. V. E. Hoggatt, Jr., & Marjorie Bicknell. "Generalized Fibonacci Polynomials and Zeckendorf's Theorem." *The Fibonacci Quarterly* 11, No. 4 (Nov. 1973):399-413.
4. V. E. Hoggatt, Jr., & Marjorie Bicknell-Johnson. "Representation of Integers in Terms of the Greatest Integer Function and the Golden Section Ratio." *The Fibonacci Quarterly* 17, No. 4 (Dec. 1979):306-17.
5. V. E. Hoggatt, Jr., & Marjorie Bicknell-Johnson. "Sequence Transforms Related to Representations Using Generalized Fibonacci Numbers." *The Fibonacci Quarterly* 20, No. 4 (Nov. 1982):289-98.

◆◇◆◇◆