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ON LUCAS POLYNOMIALS AND SOME SUMMATION FORMULAS FOR CHEBYCHEV POLYNOMIAL SEQUENCES VIA THEM

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1. INTRODUCTION

Lucas [2] defined the fundamental and the primordial functions $U_n(p, q)$ and $V_n(p, q)$, respectively, by the second-order recurrence relation

$$W_{n+2} = pW_{n+1} - qW_n \qquad (n \geqslant 0),$$

where

$$\begin{cases} \{W_n\} = \{U_n\} & \text{if } W_0 = 0,\ W_1 = 1, \text{ and} \\ \{W_n\} = \{V_n\} & \text{if } W_0 = 2,\ W_1 = p. \end{cases} \tag{1.1}$$

Let X be a matrix defined by

$$X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{1.2}$$

Taking

$$\text{tr. } X = p \qquad \text{and} \qquad \det. X = q$$

and using matrix exponential functions

$$e^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \qquad \text{and} \qquad e^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} X^{-n},$$

Barakat [1] obtained summation formulas for

$$\sum_{n=0}^{\infty} \frac{1}{n!} U_n(p, q), \ \sum_{n=0}^{\infty} \frac{1}{n!} V_n(p, q), \text{ and } \sum_{n=0}^{\infty} \frac{1}{n!} U_{n+1}(p, q).$$

Walton [7] extended Barakat's results by using the sine and cosine functions of the matrix X to obtain various other summation formulas for the functions $U_n(p, q)$ and $V_n(p, q)$. Further, using the relation between $\{U_n\}$, $\{V_n\}$, and the Chebychev polynomials $\{S_n\}$ and $\{T_n\}$ of the first and second kinds, respectively, he obtained the following summation formulas:

$$\begin{cases} \sum_{n=0}^{\infty} \frac{(-1)^n \sin 2n\theta}{(2n)!} = -\sin(\cos\theta)\sinh(\sin\theta) \\ \sum_{n=0}^{\infty} \frac{(-1)^n \sin(2n+1)\theta}{(2n+1)!} = \cos(\cos\theta)\sinh(\sin\theta) \\ \sum_{n=0}^{\infty} \frac{(-1)^n \cos 2n\theta}{(2n)!} = \cos(\cos\theta)\cosh(\sin\theta) \\ \sum_{n=0}^{\infty} \frac{(-1)^n \cos(2n+1)\theta}{(2n+1)!} = \sin(\cos\theta)\cosh(\sin\theta) \end{cases} \tag{1.3}$$

The question—Can the summation formulas for U_n and V_n and identities in (1.3) be further extended?—then naturally arises. The object of this paper is to obtain these extensions, if they exist, by using generalized circular functions.

2. GENERALIZED CIRCULAR FUNCTIONS

Pólya and Mikusiński [3] appear to be among the first few mathematicians who studied the generalized circular functions defined as follows.

For any positive integer r,

$$M_{r,j}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{rn+j}}{(rn+j)!}, \quad j = 0, 1, \ldots, r-1$$

and

$$N_{r,j}(t) = \sum_{n=0}^{\infty} \frac{t^{rn+j}}{(rn+j)!}, \quad j = 0, 1, \ldots, r-1.$$

The notation and some of the results used here are according to [4]. Note that

$$M_{1,0}(t) = e^{-t},\ M_{2,0}(t) = \cos t,\ M_{2,1}(t) = \sin t,$$

$$N_{1,0}(t) = e^{t},\ N_{2,0}(t) = \cosh t,\ N_{2,1}(t) = \sinh t.$$

Following Barakat [1] and Walton [7], we define generalized trigonometric and hyperbolic functions of any square matrix X by

$$\begin{cases} M_{r,j}(X) = \sum_{n=0}^{\infty} \frac{(-1)^n X^{rn+j}}{(rn+j)!}, & j = 0, 1, \ldots, r-1, \text{ and} \\ N_{r,j}(X) = \sum_{n=0}^{\infty} \frac{X^{rn+j}}{(rn+j)!}, & j = 0, 1, \ldots, r-1. \end{cases} \tag{2.1}$$

3. SUMMATION FORMULAS FOR THE FUNDAMENTAL FUNCTION

We use the following Lemmas.

Lemma 1: Let X be the matrix defined in (1.2), and $U_n(p, q)$ the fundamental functions defined by (1.1). Then

$$X^n = U_n X - qU_{n-1}I, \tag{3.1}$$

where I is the 2×2 unit matrix.

This lemma is proved by Barakat [1].

Lemma 2: If $f(t)$ is a polynomial of degree $\leqslant N - 1$, and if $\lambda_1, \ldots, \lambda_N$ are the N distinct eigenvalues of X, then

$$f(X) = \sum_{i=1}^{N} f(\lambda_i) \prod_{\substack{1 \leqslant j \leqslant N \\ j \neq i}} \left[\frac{X - \lambda_i I}{\lambda_i - \lambda_j}\right]. \tag{3.2}$$

This is Sylvester's matrix interpolation formula (see [6]).

Lemma 3: (a) The following identities are proved in [3]:

$$M_{3,0}(x + y) = M_{3,0}(x)M_{3,0}(y) - M_{3,1}(x)M_{3,2}(y) - M_{3,2}(x)M_{3,1}(y),$$
$$M_{3,1}(x + y) = M_{3,0}(x)M_{3,1}(y) + M_{3,1}(x)M_{3,0}(y) - M_{3,2}(x)M_{3,2}(y),$$
$$M_{3,2}(x + y) = M_{3,0}(x)M_{3,2}(y) + M_{3,1}(x)M_{3,1}(y) + M_{3,2}(x)M_{3,0}(y).$$

(b) $N_{r,j}(t) = \omega^{j/2} M_{r,j}(\omega^{-1/2} t)$, where $\omega = e^{2\pi i/r}$.

The proof is straightforward and thus omitted (for notation, see [4]).

Lemma 4: We have

$$M_{3,j}(x) - M_{3,j}(-x) = \begin{cases} -2N_{6,j+3}(x), & j = 0, 2, \\ 2N_{6,1}(x), & j = 1. \end{cases}$$

$$M_{3,j}(x) + M_{3,j}(-x) = \begin{cases} 2N_{6,j}(x), & j = 0, 2, \\ -2N_{6,4}(x), & j = 1. \end{cases}$$

Proof:

$$M_{3,j}(x) - M_{3,j}(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+j}}{(3n + j)!}[1 - (-1)^{3n+j}], \quad j = 0, 1, 2.$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+j}}{(3n + j)!}[1 - (-1)^{n+j}]$$

$$= \begin{cases} \displaystyle\sum_{n=1,3,\ldots}^{\infty} \frac{2(-1)^n x^{3n+j}}{(3n+j)!}, & j = 0, 2 \\ \displaystyle\sum_{n=0,2,\ldots}^{\infty} \frac{2(-1)^n x^{3n+j}}{(3n+j)!}, & j = 1 \end{cases}$$

$$= \begin{cases} \displaystyle -2\sum_{n=0}^{\infty} \frac{x^{6n+3+j}}{(6n+3+j)!}, & j = 0, 2 \\ \displaystyle 2\sum_{n=0}^{\infty} \frac{x^{6n+j}}{(6n+j)!}, & j = 1 \end{cases}$$

$$= \begin{cases} -2N_{6,3+j}, & j = 0, 2 \\ 2N_{6,j}, & j = 1. \end{cases}$$

The other formula can be similarly proved.

Theorem 1: The following formulas hold for $\{U_n(p, q)\}$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n}}{(3n)!} \tag{3.3}$$

$$= -\frac{2}{\delta}\{M_{3,0}(p/2)N_{6,3}(\delta/2) - M_{3,1}(p/2)N_{6,5}(\delta/2) + M_{3,2}(p/2)N_{6,1}(\delta/2)\}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n+1}}{(3n+1)!} \tag{3.4}$$

$$= \frac{2}{\delta}\{M_{3,0}(p/2)N_{6,1}(\delta/2) - M_{3,1}(p/2)N_{6,3}(\delta/2) + M_{3,2}(p/2)N_{6,5}(\delta/2)\}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n+2}}{(3n+2)!} \tag{3.5}$$

$$= -\frac{2}{\delta}\{M_{3,0}(p/2)N_{6,5}(\delta/2) - M_{3,1}(p/2)N_{6,1}(\delta/2) + M_{3,2}(p/2)N_{6,3}(\delta/2)\}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n-1}}{(3n)!} = \frac{1}{\delta q}\{\lambda_1 M_{3,0}(\lambda_1) - \lambda_2 M_{3,0}(\lambda_2)\} \tag{3.6}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n}}{(3n+1)!} = \frac{1}{\delta q}\{\lambda_1 M_{3,1}(\lambda_1) - \lambda_2 M_{3,1}(\lambda_2)\} \tag{3.7}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n+1}}{(3n+2)!} = \frac{1}{\delta q}\{\lambda_1 M_{3,2}(\lambda_1) - \lambda_2 M_{3,2}(\lambda_2)\}. \tag{3.8}$$

Here p = tr. X, q = det. X, where X is the matrix defined in (1.2) and λ_1, λ_2 are its eigenvalues. Further $\delta = \sqrt{p^2 - 4q}$.

Proof: We prove (3.4) and (3.7). The proofs of the others are similar. Since λ_1, λ_2 are the eigenvalues of X, they satisfy its characteristic equation $\lambda^2 - p\lambda + q = 0$. Therefore,

$$\lambda_1 + \lambda_2 = p, \quad \lambda_1\lambda_2 = q, \quad \text{and} \quad \lambda_1 = \frac{p + \delta}{2}, \quad \lambda_2 = \frac{p - \delta}{2}.$$

Now, using (3.2), we have

$$M_{3,1}(X) = \frac{1}{\lambda_1 - \lambda_2}\{(X - \lambda_1 I)M_{3,1}(\lambda_1) - (X - \lambda_2 I)M_{3,1}(\lambda_2)\},$$

i.e.,

$$M_{3,1}(X) = \frac{1}{\delta}\{[M_{3,1}(\lambda_1) - M_{3,1}(\lambda_2)]X - [\lambda_1 M_{3,1}(\lambda_1) - \lambda_2 M_{3,1}(\lambda_2)]I\}. \quad (3.9)$$

Using Lemma 3, we get

$$\begin{aligned} M_{3,1}(\lambda_1) - M_{3,1}(\lambda_2) &= M_{3,1}\left(\frac{p + \delta}{2}\right) - M_{3,1}\left(\frac{p - \delta}{2}\right) \\ &= M_{3,0}(p/2)[M_{3,1}(\delta/2) - M_{3,1}(-\delta/2)] \\ &\quad + M_{3,1}(p/2)[M_{3,0}(\delta/2) - M_{3,0}(-\delta/2)] \\ &\quad - M_{3,2}(p/2)[M_{3,2}(\delta/2) - M_{3,2}(-\delta/2)]. \end{aligned}$$

Now, using Lemma 4, we get

$$\begin{aligned} &M_{3,1}(\lambda_1) - M_{3,1}(\lambda_2) \quad (3.10) \\ &\quad = 2M_{3,0}(p/2)N_{6,1}(\delta/2) - 2M_{3,1}(p/2)N_{6,3}(\delta/2) + 2M_{3,2}(p/2)N_{6,5}(\delta/2). \end{aligned}$$

Substituting (3.10) in (3.9), we get

$$\begin{aligned} M_{3,1}(X) = \frac{2}{\delta}\Big\{&[M_{3,0}(p/2)N_{6,1}(\delta/2) - M_{3,1}(p/2)N_{6,3}(\delta/2) \quad (3.11) \\ &+ M_{3,2}(p/2)N_{6,5}(\delta/2)]X - \frac{1}{2}[\lambda_1 M_{3,1}(\lambda_1) - \lambda_2 M_{3,1}(\lambda_2)]I\Big\}. \end{aligned}$$

Now, by (3.1) and (2.1), we have

$$M_{3,1}(X) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(3n + 1)!}[U_{3n+1}X - qU_{3n}I]. \quad (3.12)$$

Equating the coefficients of X and I in (3.11) and (3.12), we get (3.4) and (3.7).

Starting with $M_{3,0}(X)$ and $M_{3,2}(X)$ and following a similar procedure, we obtain (3.3), (3.6), and (3.5) and (3.8).

Remark 1: The right-hand sides of (3.6)-(3.8) are expressible in terms of p and δ; however, the formulas become messy and serve no better purpose.

4. SUMMATION FORMULAS FOR THE CHEBYCHEV POLYNOMIALS

Theorem 2: The following summation formulas hold for $\{S_n(x)\}$ and $\{T_n(x)\}$. Let $x = \cos\theta$ and $y = \sin\theta$. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n S_{3n}(x)}{(3n)!} \tag{4.1}$$

$$= \frac{1}{y}[M_{3,0}(x)M_{6,3}(y) + M_{3,1}(x)M_{6,5}(y) - M_{3,2}(x)M_{6,1}(y)]$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n S_{3n+1}(x)}{(3n+1)!} \tag{4.2}$$

$$= \frac{1}{y}[M_{3,0}(x)M_{6,1}(y) + M_{3,1}(x)M_{6,3}(y) + M_{3,2}(x)M_{6,5}(y)]$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n S_{3n+2}(x)}{(3n+2)!} \tag{4.3}$$

$$= \frac{1}{y}[-M_{3,0}(x)M_{6,5}(y) + M_{3,1}(x)M_{6,1}(y) + M_{3,2}(x)M_{6,3}(y)]$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n T_{3n}(x)}{(3n)!} \tag{4.4}$$

$$= M_{3,0}(x)M_{6,0}(y) + M_{3,1}(x)M_{6,2}(y) + M_{3,2}(x)M_{6,4}(y)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n T_{3n+1}(x)}{(3n+1)!} \tag{4.5}$$

$$= -M_{3,0}(x)M_{6,4}(y) + M_{3,1}(x)M_{6,0}(y) + M_{3,2}(x)M_{6,2}(y)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n T_{3n+2}(x)}{(3n+2)!} \tag{4.6}$$

$$= -M_{3,0}(x)M_{6,2}(y) - M_{3,1}(x)M_{6,4}(y) + M_{3,2}(x)M_{6,0}(y).$$

Proof: If $p = 2x = 2\cos\theta$ and $q = 1$, then $U_n(p, q)$ are the Chebychev polynomials $S_n(x)$ of the first kind; i.e.,

$$U_n(2x, 1) = S_n(x) = \frac{\sin n\theta}{\sin\theta} \qquad (n \geqslant 0),$$

where

$$S_{n+2} = 2xS_{n+1} - S_n,$$

with

$$S_0 = 0,\ S_1 = 1.$$

We prove (4.2) and (4.5) as follows. Now,

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n+1}}{(3n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n S_{3n+1}(x)}{(3n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \sin(3n+1)\theta}{(3n+1)!\,\sin\theta}$$

$$= \frac{1}{\sin\theta} \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)!} \left[\frac{e^{i(3n+1)\theta} - e^{-i(3n+1)\theta}}{2i}\right]$$

$$= \frac{1}{2i\sin\theta} \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)!}[(e^{i\theta})^{3n+1} - (e^{-i\theta})^{3n+1}]$$

$$= \frac{1}{2i\sin\theta}[M_{3,1}(e^{i\theta}) - M_{3,1}(e^{-i\theta})].$$

Hence,

$$\sum_{n=0}^{\infty} \frac{(-1)^n U_{3n+1}}{(3n+1)!} = \frac{1}{2iy}[M_{3,1}(x+iy) - M_{3,1}(x-iy)]. \tag{4.7}$$

By Lemma 3,

$$M_{3,1}(x+iy) - M_{3,1}(x-iy) = M_{3,0}(x)[M_{3,1}(iy) - M_{3,1}(-iy)]$$
$$+ M_{3,1}(x)[M_{3,0}(iy) - M_{3,0}(-iy)]$$
$$- M_{3,2}(x)[M_{3,2}(iy) - M_{3,2}(-iy)],$$

so that by Lemma 4,

$$M_{3,1}(x+iy) - M_{3,1}(x-iy) \tag{4.8}$$
$$= 2M_{3,0}(x)N_{6,1}(iy) - 2M_{3,1}(x)N_{6,3}(iy) + 2M_{3,2}(x)N_{6,5}(iy).$$

Further, by Lemma 3(b),

$$N_{6,k}(iy) = N_{6,k}(w^{3/2}y), \text{ where } w = e^{2\pi i/6},\ k = 0, 1, \ldots, 5$$
$$= w^{k/2}M_{6,k}(wy)$$
$$= w^{k/2}\sum_{n=0}^{\infty} \frac{(-1)^n w^{6n+k}y^{6n+k}}{(6n+k)!},$$

so that

$$N_{6,k}(iy) = w^{3k/2}M_{6,k}(y). \tag{4.9}$$

Note that $w^{3/2} = i$, $w^{9/2} = -i$, and $w^{15/2} = i$. Hence, substituting (4.9) in (4.8), we get

$$\begin{aligned} &M_{3,1}(x + iy) - M_{3,1}(x - iy) \\ &\quad = 2i[M_{3,0}(x)M_{6,1}(y) + M_{3,1}(x)M_{6,3}(y) + M_{3,2}(x)M_{6,5}(y)]. \end{aligned} \tag{4.10}$$

Substituting (4.10) in (4.7), we get (4.2). It is easy to see that (4.1) and (4.3) can be similarly obtained.

Noting that

$$V_n(2x, 1) = 2T_n(x) = 2 \cos \theta,$$

and using similar techniques, we obtain (4.4)-(4.6).

Remark 2: Since $S_n(x) = \dfrac{\sin n\theta}{\sin \theta}$, and $T_n(x) = \cos n\theta$, (3.13)-(3.18) also give summation formulas for

$$\sum_{n=0}^{\infty} \frac{(-1)^n \sin (3n + j)\theta}{(3n + j)!} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1)^n \cos (3n + j)\theta}{(3n + j)!}, \; j = 0, 1, 2.$$

For example, from (3.14) we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n \sin (3n + 1)\theta}{(3n + 1)!} &= M_{3,0}(\cos \theta)M_{6,1}(\sin \theta) \\ &\quad + M_{3,1}(\cos \theta)M_{6,3}(\sin \theta) + M_{3,2}(\cos \theta)M_{6,5}(\sin \theta). \end{aligned}$$

Remark 3: Shannon and Horadam [5] studied the third-order recurrence relation

$$S_n = PS_{n-1} + QS_{n-2} + RS_{n-3} \quad (n \geqslant 4), \quad S_0 = 0,$$

where they write

$$\{S_n\} = \{J_n\} \text{ when } S_1 = 0,\ S_2 = 1, \text{ and } S_3 = P,$$

$$\{S_n\} = \{K_n\} \text{ when } S_1 = 1,\ S_2 = 0, \text{ and } S_3 = Q,$$

and

$$\{S_n\} = \{L_n\} \text{ when } S_1 = 0,\ S_2 = 0, \text{ and } S_3 = R.$$

Following Barakat, and using the matrix exponential function, they then obtained formulas for

$$\sum_{n=0}^{\infty} \frac{J_n}{n!}, \ \sum_{n=0}^{\infty} \frac{K_n}{n!}, \text{ and } \sum_{n=0}^{\infty} \frac{L_n}{n!}$$

in terms of eigenvalues of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Using matrix circular functions and their extensions and following similar techniques could be a matter of discussion for an additional paper on the derivation of the higher-order formulas for $\{J_n\}$, $\{K_n\}$, and $\{L_n\}$.

Remark 4: A question naturally arises as to whether Theorems 1 and 2 can be extended further. This encounters some difficulties, due to the peculiar behavior of $M_{r,j}(x)$ and $N_{r,j}(x)$ for higher values of r. This will be the topic of discussion in our next paper.

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