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GENERAL SOLUTION OF A FIBONACCI-LIKE RECURSION RELATION AND APPLICATIONS

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(Submitted June 1982)

1. INTRODUCTION: THE COMBINATORICS FUNCTION TECHNIQUE

In a series of papers published over the past few years, [1], [2], a method called the combinatorics function technique, or CFT, was perfected to obtain the solution of any linear partial difference equation subject to a set of initial values. Fibonacci-like recursion relations are a special case of difference equations that could be solved by the CFT method. Although many applications of the CFT have been published elsewhere, [3], [4], the study here leads to original results and provides a natural generalization of the problem investigated by Hock and McQuistan in "Occupational Degeneracy for λ-Bell Particles on a Saturated $\lambda \times N$ Lattice Space" [5].

For the reader who is not familiar with the CFT method, we summarize briefly the results of [2]. Consider a function B depending on n variables $(m_1, m_2, m_3, \ldots, m_n)$. The evaluation point, M, in the associated n-dimensional space whose coordinates are $(m_1, m_2, \ldots, m_n)$ and vector $\mathbf{M}$, whose components are the same as the coordinates of point M, will be used interchangeably for convenience. The multivariable function B is said to satisfy a partial *difference* equation when its value at point M, $B(\mathbf{M})$, is linearly related to its values at shifted arguments such as $\mathbf{M} - \mathbf{A}_k$, i.e.,

$$B(\mathbf{M}) = \sum_{k=1}^{N} f_{A_k}(\mathbf{M}) B(\mathbf{M} - \mathbf{A}_k) + I(\mathbf{M}). \tag{1.1}$$

The coefficients $f_{A_k}(\mathbf{M})$ and the inhomogeneous term $I(\mathbf{M})$ are assumed to be known and may not necessarily be constant. The problem to be investigated here is a difference equation with no inhomogeneous term, i.e.,

$$I(\mathbf{M}) = 0, \tag{1.2}$$

one-dimensional ($n = 1$), and with $N = 2$, thus corresponding to a three-term recursion relation.

The set formed by the N shifts or N displacement vectors, $\mathbf{A}_k$, is denoted $\mathscr{S}$;

$$\mathscr{A} = \{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N\}. \tag{1.3}$$

Generally, equation (1.1) holds for points M in the n-dimensional space belonging to a certain region, $\mathscr{R}$, that may not necessarily be finite. The values of a function B at a boundary, called $\mathscr{J}$, of region $\mathscr{R}$, are also generally known as

$$B(\mathbf{J}_\ell) = \Lambda_\ell;\ \ell = 1, 2, \ldots \text{ and } \mathbf{J}_\ell \in \mathscr{J}. \tag{1.4}$$

The evaluation points, $\mathbf{J}_\ell$, exhibited in equation (1.4) are referred to as boundary points for obvious reasons. Similarly, the region $\mathscr{J}$ containing the boundary points J_ℓ is interchangeably referred to as the set of boundary points or, simply, boundary set, while region $\mathscr{R}$ is the set of points M for which equation (1.1) must hold.

Equation (1.1) and its boundary value conditions, equation (1.4), were first discussed by the author and collaborators in earlier papers [1] for the special case of a single-variable function B and therefore defined on a one-dimensional space ($n = 1$). Some applications were also discussed in connection with the Schrödinger equation with a central linear potential [3]. Equation (1.1) is not necessarily consistent with the boundary value condition (1.2). To obtain the consistency condition, it was essential to introduce in [2] a set, $\mathscr{M}$, containing all points M in the n-dimensional space having the following property:

> Every possible path reaching any point belonging to set $\mathscr{M}$ by successive discrete displacement vectors $\mathbf{A}_k \in \mathscr{A}$ should contain at least one point belonging to the boundary set $\mathscr{J}$.

When such a relationship exists between two sets $\mathscr{M}$ and $\mathscr{J}$, then $\mathscr{J}$ is called a *full boundary* of $\mathscr{M}$ with respect to $\mathscr{A}$. Also essential to the consistency problem was the notion of restricted discrete paths connecting a boundary point, say $\mathbf{J}_\ell$, to a point M belonging to set $\mathscr{M}$. Such a restricted path, if it exists, does not contain any boundary point other than $\mathbf{J}_\ell$. In [2], we were able to show that there exists one and only one set, $\mathscr{J}_0$, called the *minimal full boundary* of $\mathscr{M}$ with respect to $\mathscr{A}$, such that each and every element of $\mathscr{J}_0$ can be connected to at least one element of $\mathscr{M}$ by at least one restricted path. It then follows that:

(i) equation (1.1) is consistent with equation (1.4) provided $\mathscr{R} \subset \mathscr{M}$ and,

(ii) if this is the case, its solution is unique and depends only on those ℓ's corresponding to boundary points $J_\ell \subset \mathscr{J}_0$.

The CFT method gives an explicit and systematic way of constructing the solution of equation (1.1) in terms of the so-called combinatorics functions of the *second kind*. An early version of the combinatorics functions of the second kind can be found in [1], and their applications to some physical problems are discussed in [3] and [4]. An extended and more complete version of the e functions was obtained in [2]. We now give the definition of the combinatorics functions of the second kind leading to the construction of the solution of equation (1.1).

For every boundary point $J_\ell \in \mathscr{J}_0$ and evaluation point $M \in \mathscr{M}$, one considers all possible paths connecting J_ℓ to M by discrete displacement vectors $\boldsymbol{\delta}_j \in \mathscr{A}$. A given path is identified by two labels ω and q. Label ω refers to the total number of displacements on a discrete path. Label q is used to distinguish among various paths, having the same number, ω, of displacement vectors. Corresponding to each (ωq)-path with displacement vectors $(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \ldots, \boldsymbol{\delta}_\omega)$, ordered sequentially from the boundary point J_ℓ to the evaluation point M, intermediate points, S_i, on the path are generated and represented by vectors $\mathbf{S}_i$, according to:

$$\mathbf{S}_i = \mathbf{J}_\ell + \sum_{j=1}^{i} \delta_j; \; i = 1, \ldots, \omega; \; \mathbf{S}_\omega \equiv \mathbf{M}. \tag{1.5}$$

With each (ωq)-path one associates the functional

$$F_\omega^q(\mathbf{J}_\ell ; \mathbf{M}) = \prod_{i=1}^{\omega} W(\mathbf{S}_i) f_{\delta_i}(\mathbf{S}_i). \tag{1.6}$$

Here, the f_δ's are the coefficients appearing in the difference equation (1.1), and $W(\mathbf{S}_i)$ is a weight function that may take the values 0 or 1, according to:

$$\begin{aligned} W(\mathbf{S}_i) &= 0, \text{ if } \mathbf{S}_i \in \mathscr{J}_0; \\ W(\mathbf{S}_i) &= 1, \text{ otherwise.} \end{aligned} \tag{1.7}$$

That is, F_ω^q vanishes whenever a path connecting J_ℓ to M contains an intermediate point S_i belonging to the boudary set $\mathscr{J}_0$. In other words, restricted paths are automatically selected and F_ω^q would otherwise be identically zero. The combinatorics function of the second kind associated with the boundary point J_ℓ and the evaluation point M are then defined as

$$C(\mathbf{J}_\ell;\mathbf{M}) = \sum_\omega \sum_q F_\omega^q(\mathbf{J}_\ell;\mathbf{M}). \tag{1.8}$$

Finally, the solution of a homogeneous ($I \equiv 0$) difference equation (1.1), subject to the initial conditions (1.4), when it exists, is given by

$$B(\mathbf{M}) = \sum_\ell \Lambda_\ell C(\mathbf{J}_\ell;\mathbf{M}). \tag{1.9}$$

The problem we intend to discuss is the three-term Fibonacci-like recursion relation,

$$B_m = aB_{m-1} + bB_{m-\lambda}, \tag{1.10}$$

where a and b are constant parameters and λ is a positive integer greater than unity. The case $\lambda = 2$ was discussed in detail elsewhere using the CFT (see [4]). The case $a = b = 1$ with an unspecified value of λ describes exactly the occupational degeneracy for λ-bell particles on a saturated $\lambda \times m$ lattice space when equation (1.10) is subject to a special set of initial conditions, as studied by Hock and McQuistan [5]. It is the purpose of this article to develop a unified approach to the problems of [4] and [5] that will be based on the generating function of Hock and McQuistan combined with our CFT method.

Section 2 develops the general solution of equation (1.10) subject to the unspecified initial value conditions

$$B_{-\lambda+j} = \Lambda_j \text{ for } j = 1, \ldots, \lambda - 1 \tag{1.11a}$$

$$B_0 = \Lambda_0 \tag{1.11b}$$

The choice Λ_0 for B_0 instead of Λ_λ, as might be suggested by (1.11a), is motivated by nicer looking equations appearing later in the paper.

Section 3 presents a class of generating functions that may be associated with equation (1.10). The method used there is somewhat more general than the one presented in [5].

Section 4 makes the comparison between the CFT and generating function methods, thus leading to a very interesting sum rule that is a generalization of the sum rules obtained in [4] and [5].

The conclusion of the paper is presented in Section 5.

2. THE CFT SOLUTION

The Fibonacci-like recursion relation,

$$B_m = aB_{m-1} + bB_{m-\lambda}, \ \lambda \geqslant 2, \tag{2.1}$$

will be solved for a general set of initial values:

$$\begin{aligned} B_{-\lambda+j} &= \Lambda_j, \text{ for } j = 1, \ldots, \lambda - 1, \\ B_0 &= \Lambda_0. \end{aligned} \tag{2.2}$$

This one-dimensional problem has boundary points $J_0, J_1, \ldots, J_j, \ldots, J_{\lambda-1}$ of abscissae $0, -\lambda + 1, \ldots, -\lambda + j, \ldots, -1$, respectively. The one-dimensional region, $\mathscr{M}$, consistent with the boundary region $\mathscr{J}$ consists of points M of positive integer abscissae. Paths connecting an evaluation point M of abscissa m $(m > 0)$ to a boundary point J_j of abscissa $-\lambda + j$, if $j \neq 0$, or 0, if $j = 0$, are made of displacements or steps of lengths 1 and λ. No intermediate point on these paths should belong to the boundary region $\mathscr{J}$. Boundary and evaluation points are represented on Figure 1.

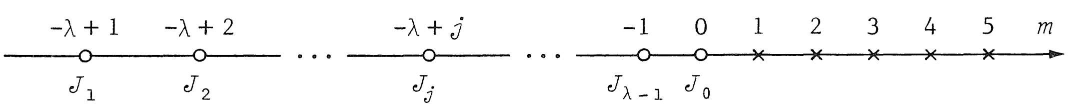

FIGURE 1

The points represented by circles "o" are boundary points and those represented by crosses "x" are evaluation points

We now proceed along the lines set by the CFT method for the construction of the combinatorics function $C(J_j\,;\,M)$. In any path, a displacement by +1 produces a factor $f_{a_1} \equiv a$ and a displacement by λ produces a factor $f_{a_2} \equiv b$. Thus, for any given path connecting J_j to M, we count the number of displacements +1 and the number of displacements λ. This path is then represented, according to the CFT method, by the product

$$F_{\omega}^{q} = a^{(\text{number of displacements } +1)} \times b^{(\text{number of displacements } \lambda)} \tag{2.3}$$

It is convenient to discuss separately the construction of $C(J_0; M)$ and $C(J_j; M)$ for $j = 1, \ldots, \lambda - 1$.

$\underline{C(J_0; M)}$

Figure 2 indicates that none of the distinct paths made of displacements +1 and λ leaving boundary point J_0 and reaching evaluation point M contains a boundary point other than J_0. In this case, none of the weight functions $W(S\ )$ is zero. Let $\left[\frac{m}{\lambda}\right]$ refer to integer divisions and $\overline{m}$ to the remainder, so that

$$m = \lambda\left[\frac{m}{\lambda}\right] + \overline{m}. \tag{2.4}$$

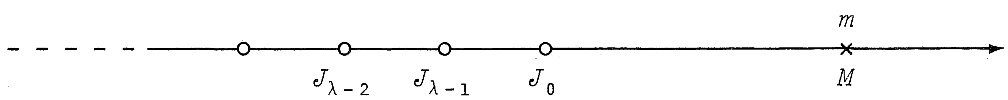

FIGURE 2

None of the distinct paths made either of displacements +1 or λ leaving J_0 and reaching the evaluation point M contains a boundary point other than J_0

Clearly, the maximum number of displacements of length λ from the origin J_0 to M is $\left[\frac{m}{\lambda}\right]$ corresponding to a minimum number of displacements of length +1 equal to $\overline{m}$. If in a path connecting J_0 to M there are k displacements of length λ $\left(0 \leqslant k \leqslant \left[\frac{m}{\lambda}\right]\right)$, then the number of displacements of length +1 must be $(m - \lambda k)$ since the length of segment J_0M is precisely equal to m. The total number of displacements in such a path is $\omega = k + (m - \lambda k)$. If q is the label of this particular path having ω displacements, then

$$F_{\omega}^{q}(J_0; M) = a^{m-\lambda k} \times b^{k}. \tag{2.5}$$

For a given total number of displacements ω, distinct paths may be generated by a reshuffling of the order of displacements of different lengths. Clearly, the distinct number of arrangements of these displacements for a given value of ω, i.e., for a given value of k $\left(0 \leqslant k \leqslant \left[\frac{m}{\lambda}\right]\right)$ is the binomial

$$q_{\max}(\omega) = \binom{m - k(\lambda - 1)}{k}. \tag{2.6}$$

Thus, the combinatorics function $C(J_0;\ M)$ is given by

$$C(J_0;\ M) = \sum_{\omega} \sum_{q=1}^{q_{\max}(\omega)} F_{\omega}^{q}(J_0;\ M)$$

$$\sum_{k=0}^{[m/\lambda]} \sum_{q=1}^{q_{\max}} a^{m-k}b^{k} = \sum_{k=0}^{[m/\lambda]} a^{m-\lambda k}b^{k}\binom{m - k(\lambda - 1)}{k} \tag{2.7}$$

<u>$C(J_j;\ M)$ for $j = 1$ to $\lambda - 1$</u>

Figure 3 shows that, in order to avoid intermeriate boundary points on any path that may connect J_j to point M, the first displacement, when leaving J_j, must be of length λ, thus reaching point M_j of abscissa j. Clearly, if the abscissa, m, of the evaluation point M is less than j, then every possible path contains at least one boundary point as an intermediate point, and the associated combinatorics function vanishes, i.e.,

$$C(J_j;\ M) \equiv 0, \text{ for } 0 < m < j. \tag{2.8}$$

On the other hand, if $m \geqslant j$, then all possible paths that contain no boundary points other than J_j have the same first displacement λ.

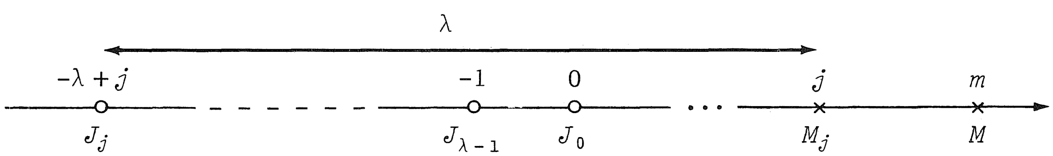

FIGURE 3

Here $j = 1$ to $\lambda - 1$. In order to avoid intermediate boundary points when leaving point J_j to reach the evaluation point M by displacements of lengths +1 and λ, the first displacement must be of length λ, thus reaching point M_j, of abscissa j

This first displacement contributes to the functional F_{ω}^{q} by producing an overall factor b. It is then straightforward to show that

$$F_{\omega}^{q}(J_j;\ M) = bF_{\omega-1}^{q}(M_j;\ M), \tag{2.9}$$

where $F_{\omega-1}(M_j;\ M)$ refers to the functional associated with the qth path cojnecting M_j to M and having $(\lambda - 1)$ dispalcements of length 1 and λ.

Since the length of segment M_jM is $m - j$, one may write an equation similar to (2.4):

$$m - j = \lambda\left[\frac{m - j}{\lambda}\right] + \overline{m - j}, \tag{2.10}$$

where the square bracket [ ], and the top bar, ——, still have the same meaning as in (2.4). The analysis for the paths connecting J_0 to M may be reproduced for the paths connecting M_j to M. The functional associated with a path having k displacements λ and $(m - j - \lambda k)$ displacements of length +1, when leaving M_j to reach M, is

$$F^q_{\omega - 1}(M_j;\, M) = a^{m-j-\lambda k} \times b^k, \tag{2.11}$$

where k may vary from 0 to $\left[\frac{m - j}{\lambda}\right]$. Combining equations (2.9) and (2.11), one finally obtains

$$\begin{aligned} C(J_j;\, M) &= \sum_{\omega}\sum_{q} bF^q_{\omega-1}(M_j;\, M) \\ &= \sum_{k=0}^{\left[\frac{m-j}{k}\right]} a^{m-j-\lambda k} \times b^{k+1}\binom{m - j - k(\lambda - 1)}{k};\ \text{for } j = 1,\ \ldots,\ \lambda - 1 \\ &\qquad\qquad \text{and } m \geqslant j. \end{aligned} \tag{2.12}$$

The domain of definition of (2.12) can easily be extended to include the region $0 < m < j$ with the understanding that the result of the operation,

$$\sum_{k=0}^{\left[\frac{m-j}{k}\right]},\ \text{for } 0 < m < j,$$

is exactly zero, so as to recover equation (2.8). With this definition in mind, the general solution of the Fibonacci-like recursion relation (2.1) subject to the boundary conditions (2.2) is

$$\begin{aligned} B_m &= \sum_{j=0}^{\lambda-1} \Lambda_j C(J_j;\, M) \\ &= \sum_{j=0}^{\lambda-1} \Lambda_j \sum_{k=0}^{\left[\frac{m-j}{\lambda}\right]} a^{m-j-k\lambda} b^{k+1-\delta_{0j}} \binom{m - j - k(\lambda - 1)}{k}. \end{aligned} \tag{2.13}$$

In (2.13), δ_{0j} is Kronecker's symbol, which is zero for $j \neq 0$, and unity otherwise.

3. THE STANDARD SOLUTION AND GENERATING FUNCTIONS

The standard solution of equation (2.1) is obtained by searching for special solutions of the form [4]:

$$B_m = R^m. \tag{3.1}$$

By requiring expression (3.1) to satisfy the recursion relation (2.1), one finds that the only possible values of R are the roots of the so-called characteristic equation, which, in this case, is of order λ; namely,

$$R^\lambda - aR^{\lambda-1} - b = 0. \tag{3.2}$$

This equation has λ roots we refer to as R_k, with index k varying from 1 to λ. The general solution of equation (2.1) is then presented in the form

$$B_m = \sum_{k=1}^{\lambda} L_k R_k^m,$$

where L_k are unspecified parameters.

Next, we will be developing a class of generating functions to the series of coefficients B_m. Following Hock and McQuistan [5], we consider functions $u_m(x)$ satisfying the recurrence relations

$$u_m(x) = A(x)u_{m-1}(x) + B(x)u_{m-\lambda}(x). \tag{3.4}$$

$A(x)$ and $B(x)$ are some chosen functions of x restricted to take on the values

$$A(x_0) = a, \qquad B(x_0) = b, \tag{3.5}$$

when the variable $x = x_0$. Furthermore, for any value of the variable x, one also requires

$$\begin{aligned} u_{-\lambda+j}(x) &\equiv \Lambda_j, \text{ for } j = 1, \ldots, \lambda - 1 \\ u_0(x) &\equiv \Lambda_0. \end{aligned} \tag{3.6}$$

Clearly, $u_m(x)$ evaluated at $x = x_0$ is precisely B_m, whose explicit expression was obtained via the CFT method, namely, equation (2.13). The bivariant generating function $u(x, y)$ is given by [5]:

$$u(x, y) = \sum_{m=0}^{\infty} u_m(x)y^m. \tag{3.7}$$

One separates the summation over m into two pieces as follows:

$$u(x,\ y) = \sum_{m=0}^{\lambda-1} u_m(x)y^m + \sum_{m=\lambda}^{\infty} u_m(x)y^m. \tag{3.8}$$

Next, one replaces $u_m(x)$ appearing in the second summation by the right-hand side of the recurrence relation (3.4),

$$\sum_{m=\lambda}^{\infty} u_m(x)y^m = \sum_{m=\lambda}^{\infty} A(x)u_{m-1}(x)y^m + \sum_{m=\lambda}^{\infty} B(x)u_{m-\lambda}(x)y^m. \tag{3.9}$$

It is straightforward to recognize that

$$\begin{aligned}\sum_{m=1}^{\infty} A(x)u_{m-1}(x)y^m &= \sum_{m=1}^{\infty} A(x)u_{m-1}(x)u_{m-1}(x)y^m - \sum_{m=1}^{\lambda-1} A(x)u_{m-1}(x)y^m \\ &= A(x)y^m u(x,\ y) - \sum_{m=1}^{\lambda-1} A(x)u_{m-1}(x)y^m,\end{aligned} \tag{3.10}$$

and, also, that

$$\sum_{m=\lambda}^{\infty} B(x)u_{m-\lambda}(x)y^m = B(x)y^{\lambda}u(x,\ y). \tag{3.11}$$

Combining (3.8), (3.9), and (3.10) and (3.11), one finds

$$u(x,\ y)\ [1 - A(x)y - B(x)y^{\lambda}] = \sum_{m=0}^{\lambda-1} u_m(x)y^m - \sum_{m=1}^{\lambda-1} A(x)u_{m-1}(x)y^m. \tag{3.12}$$

A last manipulation on the right-hand side of equation (3.12) is possible to obtain the explicit form of the bivariant generating function $u(x,\ y)$; namely,

$$\sum_{m=1}^{\lambda-1} u_m(x)y^m - \sum_{m=1}^{\lambda-1} A(x)u_{m-1}(x)y^m = u_0(x) + \sum_{m=1}^{\lambda-1} [u_m(x) - A(x)u_{m-1}(x)]y^m. \tag{3.13}$$

This is followed by the use of the recurrence relation (3.4) and its initial conditions (3.6):

$$\sum_{m=1}^{\lambda-1} [u_m(x) - A(x)u_{m-1}(x)] = \sum_{m=1}^{\lambda-1} B(x)u_{m-\lambda}(x)y^m = \sum_{m-1}^{\lambda-1} B(x)\Lambda_m y^m. \tag{3.14}$$

Finally, combining equations (3.12), (3.13), (3.14), and, again, (3.6), one finds

$$u(x,\ y) = \frac{\Lambda_0 + B(x)\sum_{j=1}^{\lambda-1}\Lambda_j y^j}{1 - A(x)y + B(x)y^{\lambda}}. \tag{3.15}$$

Equation (3.15) is a generalization of the bivariant generating function of [5], where $\Lambda_j = 0$ for $j = 1$ to $\lambda - 1$, $\Lambda_0 = 1$, $A(x) = 1$, and $B(x) = x$.

4. SUM RULES

The two approaches presented in Sections 2 and 3 must be equivalent. By making use of this equivalence, a sum rule will naturally emerge. One sets $x = x_0$ in equations (3.7) and (3.15), and takes into account the restrictions $A(x_0) = a$, $B(x_0) = b$, and the property $u_m(x_0) = B_m$. It then follows that

$$\sum_{m=0}^{\infty} B_m y^m \equiv \frac{\Lambda_0 + b\sum_{j=1}^{\lambda-1}\Lambda_j y^j}{1 - ay - by^\lambda} = \frac{\sum_{j=0}^{\lambda-1} b^{1-\delta_{0j}}\Lambda_j y^j}{1 - ay - by^\lambda} \tag{4.1}$$

Let

$$f(y) = 1 - ay - by^\lambda. \tag{4.2}$$

This polynomial is of order λ; it has λ roots we call y_k, $k = 1, \ldots, \lambda$. When comparing equations (4.2) and (3.2), it is evident that the roots R_k of equation (3.2) are the inverses of the roots y_k of equation (4.2), i.e.,

$$y_k = \frac{1}{R_k}. \tag{4.3}$$

The standard expension of $\dfrac{1}{f(y)}$ in terms of the zeros of the function $f(y)$ is

$$\frac{1}{f(y)} = \sum_{k=1}^{\lambda} \frac{D_k}{y - y_k} = -\sum_{k=1}^{\lambda} \frac{(D_k R_k)}{1 - yR_k}, \tag{4.4}$$

and the residue D_k is obtained in the usual manner as

$$D_k = \lim_{y \to y_k} \frac{y - y_k}{f(y)} = -\frac{1}{f'(y_k)} = -\frac{1}{a + b\lambda R_k^{1-\lambda}} \tag{4.5}$$

Thus, equation (4.1) yields

$$\sum_{m=0}^{\infty} B_m y^m \equiv \sum_{j=0}^{\lambda-1} b^{1-\delta_{0j}}\Lambda_j y^j \sum_{k=1}^{\lambda} \frac{R_k}{a + b\lambda R_k^{1-\lambda}} \quad \frac{1}{1 - yR_k} \tag{4.6}$$

The left-hand side (lhs) of equation (4.6) contains B_m whose explicit dependence on Λ_j has been derived in Section 2, equation (2.13). It can be written as

$$(\text{lhs}) = \sum_{j=0}^{\lambda-1} \Lambda_j \sum_{m=0}^{\infty} y^m \sum_{k=0}^{\left[\frac{m-j}{\lambda}\right]} a^{m-j-k\lambda} b^{k+1-\delta_{j0}} \binom{m-j-k(\lambda-1)}{k}. \tag{4.7}$$

The right-hand side (rhs) of equation (4.6) can be rearranged using the power series expansion of $(1 - yR_k)^{-1}$ to yield

$$(\text{rhs}) = \sum_{j=0}^{\lambda-1} b^{1-\delta_{0j}} \Lambda_j \sum_{m=0}^{\infty} y^{j+m} \sum_{k=1}^{\lambda} \frac{R_k^{m-1}}{a + b\lambda R_k^{1-\lambda}} \tag{4.8}$$

Recalling that $\sum_{k=0}^{\left[\frac{m-j}{\lambda}\right]} \equiv 0$ for $0 < m < j$, then

$$(\text{lhs}) = \sum_{j=0}^{\lambda-1} \Lambda\ b^{1-\delta_{0j}} \sum_{m=j}^{\infty} y \sum_{k=0}^{\left[\frac{m-j}{\lambda}\right]} a^{m-j-k\lambda} b^k \binom{m-j-k(\lambda-1)}{k}. \tag{4.9}$$

Also, making the shift $m \to m - j$ in the summation index m of (4.8), one has

$$(\text{rhs}) = \sum_{j=0}^{\lambda-1} b^{1-\delta_{0j}} \Lambda_j \sum_{m=j}^{\infty} y \sum_{k=1} \frac{R_k^{m+1-j}}{a + b\lambda R_k^{1-\lambda}}. \tag{4.10}$$

Clearly, since (lhs) and (rhs) are equivalent expressions, and, since Λ_j are completely arbitrary parameters, one necessarily has the identity:

$$\sum_{k=0}^{\left[\frac{m-j}{\lambda}\right]} a^{m-k\lambda} b^k \binom{m-j-k(\lambda-1)}{k} \equiv \sum_{k=1}^{\lambda} \frac{R_k^{m+1-j}}{a + b\lambda R_k^{1-\lambda}}, \tag{4.11}$$

which holds for $m \geqslant j \geqslant 0$, or, simply,

$$\sum_{k=0}^{[m/\lambda]} a^{m-k\lambda} b^k \binom{m-k(\lambda-1)}{k} \equiv \sum_{k=1}^{\lambda} \frac{R_k^{m-1}}{a + b\lambda R_k^{1-\lambda}}, \text{ for } m \geqslant 0. \tag{4.12}$$

This identity reproduces the one derived by Hock and McQuistan [5] for $a = b = 1$ and any value of λ, and the sum rule derived by Phares and Simmons [4] for $\lambda = 2$ and arbitrary values of a and b. Indeed, for $\lambda = 2$, the two roots R_1 and R_2 of equations (3.2) are (see [4]):

$$\begin{aligned} R_1 &= (1/2)[a + (a^2 + 4b)^{\frac{1}{2}}], \\ R_2 &= (1/2)[a - (a^2 + 4b)^{\frac{1}{2}}]. \end{aligned} \tag{4.13}$$

It is then easy to check that the left-hand side of equation (4.11), with $\lambda = 2$, and, the values of R_1 and R_2 given by equation (4.13), becomes

$$\sum_{k=1}^{2} \frac{R_k^{m+1}}{a + 2bR_k^{-1}} = \frac{[a + (a^2 + 4b)^{\frac{1}{2}}] - [a - (a^2 + 4b)^{\frac{1}{2}}]^{m+1}}{2^{m+1}(a^2 + 4b)^{\frac{1}{2}}}. \tag{4.14}$$

Equation (4.14) shows that equation (4.12) reduces, for $\lambda = 2$, to equation (4.10) of [4].

5. CONCLUSION

A Fibonacci recurrence relation with constant coefficients has been solved exactly for arbitrary initial conditions using the combinatorics function techniques. A class of generating functions involving two arbitrarily chosen functions $A(x)$ and $B(x)$ has been obtained. The method of Hock and McQuistan applied to the generating function and combined with the CFT solution leads to a sum rule that reproduces the two special cases discussed in [4] and [5].

The flexibility shown in the application of the CFT method in the simple case presented here is not an exception. More complicated problems have been solved involving two-dimensional, homogeneous, three-term difference equations with variable coefficients. The interested reader may refer to [6].

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