MULTIPARAMETER STIRLING AND C-NUMBERS: RECURRENCES AND APPLICATIONS

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1. INTRODUCTION

Stirling numbers of the first and second kind are less known among statisticians than among people who deal with combinatorics or finite differences. Only recently have they made their appearance in distribution theory and statistics. They emerge in the distribution of a sum of zero-truncated classical discrete distributions: those of the second kind, $S(m, n)$, in the case of a Poisson distribution truncated away from zero, Tate & Goen [13], Cacoullos [2], the signless (absolute-value) Stirling numbers of the first kind, $|s(m, n)|$, in the logarithmic series distribution, Patil [9]. In general, such distributional problems are essential in the construction of minimum variance unbiased estimators (mvue) for parametric functions of a left-truncated power series distribution (PSD).

Analogous considerations for binomial and negative binomial distributions truncated away from zero motivated the introduction of a new kind of numbers, called C-numbers by Cacoullos & Charalambides [5]. These three-parameter C-numbers, $C(m, n, k)$, were further studied by Charalambides [8], who gave the representation

$$C(m, n, k) = \sum_{r=n}^{m} k^r s(m, r)S(r, n)$$

in terms of Stirling numbers of the first kind, $s(m, r)$, and the second kind, $S(r, n)$. Interestingly enough, this representation in a disguised form was, in effect, used by Shumway & Gurland [11] to tabulate C-numbers, involved in the calculation of Poisson-binomial probabilities.

The so-called generalized Stirling and C-numbers emerged as a natural extension of the corresponding simple ones in the study of the mvue problem for a PSD truncated on the left at an arbitrary (known or unknown) point (Charalambides [7]). It should be mentioned that, in particular, the generalized Stirling numbers of the second kind were independently rediscovered and tabulated by Sobel *et al.* [12], in connection with the Incomplete Type I-Dirichelt integral.

The multiparameter Stirling and C-numbers are the analogues of generalized Stirling and C-numbers in a multi-sample situation where the underlying PSD is multiply truncated on the left (Cacoullos [3], [4]).

Recurrence relations for ratios of Stirling and C-numbers are necessary, because the mvue of certain parametric functions of left-truncated logarithmic series, Poisson, binomial and negative binomial distributions are expressed in terms of such ratios. These recurrences bypass the computational difficulties which come from the fact that the numbers themselves (but not the ratios of interest) grow very fast with increasing arguments. Recurrences for ratios of simple Stirling numbers of the second kind were developed by Berg [1].

The main purpose of this paper is to provide recurrences for certain ratios of multiparameter Stirling and C-numbers, thus unifying several special results, including those of Berg [1]. For the development of the topic, we found the use of exponential generating functions (egf) most appropriate both for introducing the numbers themselves and for deriving recurrences. Without claiming completeness, we included certain basic recurrences. As observed elsewhere, Cacoullos [3] and [4], it is emphasized here, once more, that in the study of PSDs the egf approach is the one suggested by the probability function itself in its truncated form. Also, we found it appropriate to include certain asymptotic relations between Stirling and C-numbers, which reflect corresponding relations between binomial and Poisson distributions or logarithmic series and negative binomial distributions.

A typical result, which involves ratios considered here, is the following: Let x_{ij}, $j = 1, \ldots, n_i$, be a random sample from a left-truncated one-parameter PSD distribution with p.f.

$$p(x;\,\theta) = \frac{a_i(x)\theta^x}{f_i(\theta,\,r_i)},\quad x = r_i,\ r_i + 1,\ \ldots, \tag{1.1}$$

where

$$f_i(\theta,\,r_i) = \sum_{x=r_i}^{\infty} a_i(x)\theta^x,\quad i = 1,\ \ldots,\ k.$$

If the truncation point $\underset{\sim}{r} = (r_1, \ldots, r_k)$ is known and $a_i(x) > 0$ for every $x > r_i$, $i = 1, \ldots, k$, then, according to Cacoullos [4], for every $j = 1, 2, \ldots$, θ_j is estimable and its (unique) mvue, based on all k independent samples $\{x_{ij}\}$, is given by

$$\hat{\theta}_j(m) = (m)_j\,\frac{a(m - j;\,\underset{\sim}{n},\,\underset{\sim}{r})}{a(m;\,\underset{\sim}{n},\,\underset{\sim}{r})}, \tag{1.2}$$

where $\underset{\sim}{n} = (n_1, \ldots, n_k)$, $\underset{\sim}{r} = (r_1, \ldots, r_k)$, $(m)_j = m(m-1)\cdots(m-j+1)$ and

$$a(m;\,\underset{\sim}{n},\,\underset{\sim}{r}) = \frac{m!}{n_1!\,\ldots\,n_k!}\sum_m \prod_{i=1}^{k}\prod_{j=1}^{n_i} a_i(x_{ij}), \tag{1.3}$$

where the summation extends over all ordered N-tuples $(N = n_1 + \cdots + n_k)$ of integers x_{ij} satisfying $x_{ij} \geqslant r_i$,

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i} x_{ij} = m.$$

In the cases of interest (Poisson, binomial, and so on), the numbers (integers) $a(m;\,\underset{\sim}{n},\,\underset{\sim}{r})$ turn out to be Stirling or C-numbers, depending on the series function f_i in (1.1), which at the same time suggests the corresponding egf of these numbers.

2. MULTIPARAMETER STIRLING NUMBERS OF THE FIRST KIND: DEFINITION-GENERAL PROPERTIES

Let $r_1, \ldots, r_k$ and $n_1, \ldots, n_k$ be nonnegative integers ($k \geqslant 1$). The multiparameter Stirling numbers of the first kind with parameters $\underset{\sim}{r} = (r_1, r_2, \ldots, r_k)$ and $\underset{\sim}{n} = (n_1, n_2, \ldots, n_k)$, to be denoted by $s(m;\,\underset{\sim}{n},\,\underset{\sim}{r})$, can be defined (cf. Cacoullos [3]) by the egf,

$$g_{\underset{\sim}{n}}(t;\ \underset{\sim}{r}) = \sum_{m=\underset{\sim}{r}'\underset{\sim}{n}}^{\infty} s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})t^m/m!$$
$$= \prod_{i=1}^{k} \frac{1}{n_i!}\left[\log(1 + t) - \sum_{j=1}^{r_i - 1} (-1)^{j-1}\frac{t^j}{j}\right]^{n_i}, \tag{2.1}$$

where we set $m = \underset{\sim}{r}'\underset{\sim}{n} = r_1 n_1 + \cdots + r_k n_k$.

The special case $k = 1$, $r_1 = r$, $n_1 = n$ yields the generalized Stirling numbers of the first kind, $s(m;\ n,\ r)$, defined by Charalambides [6], while $k = 1$, $r = 1$ gives the simple Stirling numbers of the first kind, $s(m,\ n)$. Propositions 2.1 - 2.3 summarize basic properties and recurrences for $s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})$ and facilitate their computation.

Remark 2.1: In the sequel, in order to avoid unnecessary complications in the recurrences, we assume that all $n_i > 0$, some n_i, say ν, are zero, then the parameter k becomes $k' = k - \nu$ and the necessary modifications are obvious.

Proposition 2.1: The multiparameter Stirling numbers of the first kind $s(m;\underset{\sim}{n},\underset{\sim}{r})$ have the following representation

$$s(m;\ \underset{\sim}{n},\ r) = (-1)^{m-N}\frac{m!}{n_1!\ \ldots\ n_k!}\sum_m \prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{1}{x_{ij}}, \tag{2.2}$$

where $N = n_1 + \cdots + n_k$ and the summation extends over all ordered N-tuples of integers x_{ij} satisfying the relations

$$x_{ij} \geqslant r_i,\ i = 1,\ \ldots,\ k \quad \text{and} \quad \sum_{i=1}^{k}\sum_{j=1}^{n_i} x_{ij} = m.$$

Proof: We have

$$\gamma(t,\ r_i) = \log(1 + t) - \sum_{k=1}^{r_i - 1}(-1)^{k-1}\frac{t^k}{k}$$
$$= \sum_{k=r_i}^{\infty}(-1)^{k-1}\frac{t^k}{k},\ i = 1,\ \ldots,\ k. \tag{2.3}$$

Forming the Cauchy product of series, we find, by virtue of (2.1),

$$g_{\underset{\sim}{n}}(t;\ \underset{\sim}{r})\prod_{i=1}^{k} n_i! = \prod_{i=1}^{k}[\gamma(t,\ r_i)]^{n_i}$$
$$= \sum_{m=\underset{\sim}{r}'\underset{\sim}{n}}^{\infty}(-1)^{m-N}t^m\sum_m\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{1}{x_{ij}}, \tag{2.4}$$

where $\sum_m$ has the same meaning as above. Comparing (2.4) with (2.1), we get (2.2).

To obtain recurrence relations, we make use of the easily verified difference/differential equation, satisfied by the egf $g_{\underset{\sim}{n}}(t;\ \underset{\sim}{r})$ in (2.1), namely,

$$(1 + t)\frac{d}{dt}g_{\underset{\sim}{n}}(t;\ \underset{\sim}{r}) = \sum_{i=1}^{k}(-1)^{r_i - 1}t^{r_i - 1}g_{\underset{\sim}{n} - \underset{\sim}{e}_i}(t;\ \underset{\sim}{r}), \tag{2.5}$$

where $\underset{\sim}{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, i.e., a k-component vector with zero components except the i^{th} component, which is equal to 1.

Proposition 2.2: $(m; \underset{\sim}{n})$-wise relations: The numbers $s(m; \underset{\sim}{n}, \underset{\sim}{r})$ satisfy the recurrence relation

$$s(m+1; \underset{\sim}{n}, \underset{\sim}{r}) + ms(m; \underset{\sim}{n}, \underset{\sim}{r})$$

$$= \sum_{i=1}^{k} (-1)^{r_i - 1} (m)_{r_i - 1}\, s(m - r_i + 1; \underset{\sim}{n} - \underset{\sim}{e}_i, \underset{\sim}{r}) \tag{2.6}$$

with initial conditions

$$s(0; 0, \underset{\sim}{r}) = 1, \; s(0; \underset{\sim}{n}, \underset{\sim}{r}) = 0 \text{ whenever } \sum_{i=1}^{k} r_i n_i > 0,$$

$$s(m; \underset{\sim}{n}, \underset{\sim}{r}) = 0 \text{ if } m < \underset{\sim}{r}'\underset{\sim}{n}.$$

Proof: Equation (2.5) by virtue of (2.1) can be written as

$$(1+t) \sum_{m = \underset{\sim}{r}'\underset{\sim}{n}}^{\infty} s(m; \underset{\sim}{n}, \underset{\sim}{r}) \frac{t^{m-1}}{(m-1)!}$$

$$= \sum_{i=1}^{k} \sum_{m = \underset{\sim}{r}'\underset{\sim}{n} - r_i}^{\infty} (-1)^{r_i - 1} s(m; \underset{\sim}{n} - \underset{\sim}{e}_i, \underset{\sim}{r}) \frac{t^{m + r_i - 1}}{m!}. \tag{2.7}$$

Equating the coefficients of $t^m/m!$ in (2.7) yields (2.6). Note that equation (2.6) for $k = 1$, $r_1 = 1$ gives the well-known recurrence for the simple Stirling numbers of the first kind

$$s(m+1, n) = s(m, n-1) - ms(m, n). \tag{2.8}$$

Proposition 2.3: $(m; \underset{\sim}{n}, \underset{\sim}{r})$-wise relations: The numbers $s(m; \underset{\sim}{n}, \underset{\sim}{r})$ satisfy

$$s(m; \underset{\sim}{n}, \underset{\sim}{r} + \underset{\sim}{e}_i) = \sum_{j=0}^{n_i} (-1)^{j r_i} \frac{(m)_{jr}}{j!\,(r_i)^j}\, s(m - jr_i; \underset{\sim}{n} - j\underset{\sim}{e}_i, \underset{\sim}{r}), \; i = 1, \ldots, k. \tag{2.9}$$

Proof: We have, using also (2.3),

$$g_{\underset{\sim}{n}}(t; \underset{\sim}{r} + \underset{\sim}{e}_i) = \frac{1}{n_i!}\left[\gamma(t; \underset{\sim}{r}) + (-1)^{r_i} \frac{t^{r_i}}{r_i}\right]^{n_i} \prod_{\substack{j \neq i \\ j=1}}^{k} \frac{1}{n_j!}[\gamma(t, r_j)]^{n_j}; \tag{2.10}$$

and using the binomial expansion

$$\left[\gamma(t, r_i) + (-1)^{r_i} \frac{t^{r_i}}{r_i}\right]^{n_i} = \sum_{j=0}^{n_i} \binom{n_i}{j} (n_i - j)!\, g_{n_j - j}(t, r_i) (-1)^{j r_i} \frac{t^{j r_i}}{r_i^j}, \tag{2.11}$$

we can write (2.10) as

$$\sum_{m = \underset{\sim}{r}'\underset{\sim}{n} + n_i}^{\infty} s(m; \underset{\sim}{n}, \underset{\sim}{r} + \underset{\sim}{e}_i) \frac{t^m}{m!} = \sum_{j=0}^{n_i} \frac{(-1)^{j r_i}}{j!\, r_i^j} \sum_{m = \underset{\sim}{r}'\underset{\sim}{n} - j r_i} s(m; \underset{\sim}{n} - j\underset{\sim}{e}_i, \underset{\sim}{r}) \frac{t^{m + j r_i}}{m!}. \tag{2.12}$$

Hence, equating the coefficients of $t^m/m!$, we obtain (2.9).

Signless Multiparameter Stirling Numbers

From the recurrence relation (2.6), it follows that the numbers $s(m;\, \underset{\sim}{n},\, \underset{\sim}{r})$ are integers. Moreover, from the representation in (2.2), we conclude that $s(m;\, \underset{\sim}{n},\, \underset{\sim}{r})$ is an integer with sign $(-1)^{N-m}$, where $N = n_1 + \cdots + n_k$. Therefore, if we multiply (2.6) by $(-1)^{m-N+1}$, we obtain

$$|s(m+1;\, \underset{\sim}{n},\, \underset{\sim}{r})| = m|s(m;\, \underset{\sim}{n},\, \underset{\sim}{r})| + \sum_{i=1}^{k} (m)_{r_i - 1} |s(m - r_i + 1;\, \underset{\sim}{n} - \underset{\sim}{e}_i,\, \underset{\sim}{r})|. \tag{2.13}$$

We call $|s(m;\, n,\, r)|$ the signless (positive) multiparameter (k-parameter)

Stirling Number of the First Kind. We will show

Proposition 2.4: The egf of $|s(m;\, \underset{\sim}{n},\, \underset{\sim}{r})|$ is given by

$$g_{\underset{\sim}{n}}^{*}(t;\, \underset{\sim}{r}) = \sum_{m=\underset{\sim}{r}'\underset{\sim}{n}}^{\infty} |s(m;\, \underset{\sim}{n},\, \underset{\sim}{r})| \frac{t^m}{m!} = \prod_{i=1}^{k} \frac{1}{n_i!}\left[-\log(1 - t) - \sum_{j=1}^{r_i - 1} \frac{t^j}{j}\right]^{n_i}. \tag{2.14}$$

Proof: From the difference equation (2.13), it is easily verified that the egf $g_{\underset{\sim}{n}}^{*}(t;\, \underset{\sim}{r})$ satisfies the difference/differential equation

$$(1 - t)\frac{d}{dt} h_{\underset{\sim}{n}}(t;\, \underset{\sim}{r}) = \sum_{i=1}^{k} t^{r_i - 1} g_{\underset{\sim}{n} - \underset{\sim}{e}_i}^{*}(t;\, \underset{\sim}{r}), \tag{2.15}$$

which, in turn, yields (2.14).

Alternatively, (2.14) leads to the representation of $|s(m;\, \underset{\sim}{n},\, \underset{\sim}{r})|$ as obtained from (2.2).

3. RATIOS OF MULTIPARAMETER STIRLING NUMBERS OF THE FIRST KIND

We define, as a ratio of multiparameter Stirling numbers of the first kind with respect to argument m, the function

$$R_1(m;\, \underset{\sim}{n},\, \underset{\sim}{r}) = \frac{s(m + 1;\, \underset{\sim}{n},\, \underset{\sim}{r})}{s(m;\, \underset{\sim}{n},\, \underset{\sim}{r})}. \tag{3.1}$$

Ratios with respect to the arguments n_i, r_i, $i = 1, \ldots, k$, can also be defined The main reason for considering ratios with respect to m is seen from (1.1), which actually involves reciprocals of R_1, when we are concerned with the parameter of a logarithmic series distribution.

Proposition 3.1: A recurrence relation for the ratio $R_1(m;\, \underset{\sim}{n},\, \underset{\sim}{r})$, independent of the multiparameter Stirling numbers of the first kind, is given by

$$R_1(m;\, \underset{\sim}{n},\, \underset{\sim}{r}) + m = \frac{\displaystyle\sum_{j=1}^{k} \frac{(m)_{r_j - 1} r_j n_j}{(\underset{\sim}{r}'\underset{\sim}{n})r_j} \prod_{i=1}^{m+1-\underset{\sim}{r}'\underset{\sim}{n}} R_1(m - r_j + 1 - i;\, \underset{\sim}{n} - \underset{\sim}{e}_j,\, \underset{\sim}{r})}{\displaystyle\prod_{i=1}^{m-\underset{\sim}{r}'\underset{\sim}{n}} R_1(m - i;\, \underset{\sim}{n},\, \underset{\sim}{r})} \tag{3.2}$$

for $\underset{\sim}{n} > \underset{\sim}{1}$ and $m > \underset{\sim}{r}'\underset{\sim}{n}$, with the boundary conditions

$$R_1(m,\ 1,\ r) = -m \tag{3.3}$$

and

$$R_1(\underset{\sim}{r}'\underset{\sim}{n};\ \underset{\sim}{n},\ \underset{\sim}{r}) = (-1)(\underset{\sim}{r}'\underset{\sim}{n} + 1)\sum_{j=1}^{k}\frac{r_j n_j}{(r_j + 1)} \tag{3.4}$$

Proof: Using equation (3.1), it can easily be seen that

$$\prod_{i=1}^{m-\underset{\sim}{r}'\underset{\sim}{n}} R_1(m - i;\ \underset{\sim}{n},\ \underset{\sim}{r}) = \frac{s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})}{s(\underset{\sim}{r}'\underset{\sim}{n};\ \underset{\sim}{n},\ \underset{\sim}{r})}. \tag{3.5}$$

But equation (2.2), for $m = \underset{\sim}{r}'\underset{\sim}{n}$, $m_{i1} = m_{i2} = \cdots = m_{in_i} = r_i$, becomes

$$s(\underset{\sim}{r}'\underset{\sim}{n};\ \underset{\sim}{n},\ \underset{\sim}{r}) = (-1)^{\underset{\sim}{r}'\underset{\sim}{n}-N}\frac{(\underset{\sim}{r}'\underset{\sim}{n})!}{\prod\limits_{i=1}^{k} n_i!\ \prod\limits_{i=1}^{k} r_i^{n_i}}. \tag{3.6}$$

Consequently, equation (3.5) becomes

$$s(m;\ \underset{\sim}{n},\ \underset{\sim}{r}) = \frac{(-1)^{\underset{\sim}{r}'\underset{\sim}{n}-N}(\underset{\sim}{r}'\underset{\sim}{n})!\ \prod\limits_{i=1}^{m-\underset{\sim}{r}'\underset{\sim}{n}} R_1(m - i;\ \underset{\sim}{n},\ \underset{\sim}{r})}{\prod\limits_{i=1}^{k} n_i!\ \prod\limits_{i=1}^{k} r_i^{n_i}} \tag{3.7}$$

From equations (2.6) and (3.1) we have

$$R_1(m;\ \underset{\sim}{n},\ \underset{\sim}{r}) + m = \frac{\sum\limits_{j=1}^{k}(-1)^{r_j - 1}(m)_{r_j - 1}\, s(m - r_j + 1;\ \underset{\sim}{n} - \underset{\sim}{e}_j,\ \underset{\sim}{r})}{s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})} \tag{3.8}$$

and substituting for $s(m - r_j + 1;\ \underset{\sim}{n} - \underset{\sim}{e}_j,\ \underset{\sim}{r})$ and $s(m;\ \underset{\sim}{n},\ \underset{\sim}{r})$ from (3.7) yields (3.2). By definition,

$$R_1(\underset{\sim}{r}'\underset{\sim}{n};\ \underset{\sim}{n},\ \underset{\sim}{r}) = \frac{s(\underset{\sim}{r}'\underset{\sim}{n} + 1;\ \underset{\sim}{n},\ \underset{\sim}{r})}{s(\underset{\sim}{r}'\underset{\sim}{n};\ \underset{\sim}{n},\ \underset{\sim}{r})}, \tag{3.9}$$

and since equation (2.2), for $m = \underset{\sim}{r}'\underset{\sim}{n} + 1$, becomes

$$s(\underset{\sim}{r}'\underset{\sim}{n} + 1;\ \underset{\sim}{n},\ \underset{\sim}{r}) = (-1)^{\underset{\sim}{r}'\underset{\sim}{n}+1-N}\frac{(\underset{\sim}{r}'\underset{\sim}{n} + 1)!}{\prod\limits_{\substack{i=1\\ j\neq i}}^{k} n_i!}\sum_{i=1}^{k}\frac{n_j}{r_j^{n_j - 1}(r_j + 1)\prod\limits_{i\neq 1}^{k} r_i^{n_i}}, \tag{3.10}$$

by using equation (3.6), the required formula (3.4) is easily obtained.

The special case $k = 1$ yields

Proposition 3.2: A recurrence relation for the ratio $R_1(m,\ n,\ r)$, independent of the generalized Stirling numbers of the first kind, is given by

$$R_1(m, n, r) + m = \frac{\dfrac{rn(m)_{r-1}}{(rn)_r} \displaystyle\prod_{i=1}^{m+1-rn} R_1(m - r + 1 - i, n - 1, r)}{\displaystyle\prod_{i=1}^{m-rn} R_1(m - i, n, r)} \tag{3.11}$$

for $n > 1$ and $m > rn$, with

$$R_1(m, 1, r) = -m \tag{3.12}$$

$$R_1(rn, n, r) = -\frac{rn(rn + 1)}{r + 1} \tag{3.13}$$

Also, for $k = 1$, $r = 1$, we obtain

Proposition 3.3: A recurrence relation for the ratio $R_1(m, n)$, independent of the simple Stirling numbers of the first kind, is given by

$$R_1(m, n) + m = \frac{\displaystyle\prod_{i=1}^{m+1-n} R_1(m - i, n - 1)}{\displaystyle\prod_{i=1}^{m-n} R_1(m - i, n)} \tag{3.14}$$

for $n > 1$ and $m > n$, with

$$R_1(m, 1) = -m \tag{3.15}$$

$$R_1(n, n) = -n(n + 1)/2 \tag{3.16}$$

Proposition 3.4: An alternative recurrence relation for the ratio $R_1(m, n, r)$ is given by

$$R_1(m, n, r) + m = \frac{m}{m - r + 1} \frac{[R_1(m - 1, n, r) + m - 1]R_1(m - r, n - 1, r)}{R_1(m - 1, n, r)} \tag{3.17}$$

for $n > 1$ and $m > rn$. $R_1(m, 1, r)$ and $R_1(rn, n, r)$ are given by (3.12) and (3.13), respectively.

Proof: Using equation (2.6) with $k = 1$, we have

$$R_1(m, n, r) + m = \frac{(-1)^{r-1}(m)_{r-1}s(m - r + 1, n - 1, r)}{s(m, n, r)} \tag{3.18}$$

from which equation (3.17) can easily be derived.

Applying Proposition 3.4 with $r = 1$ gives

Proposition 3.5: An alternative recurrence relation for the ratio $R_1(m, n)$ is given by

$$R_1(m, n) + m = \frac{[R_1(m - 1, n) + m - 1]R_1(m - 1, n - 1)}{R_1(m - 1, n)} \tag{3.19}$$

for $n > 1$ and $m > n$.

4. MULTIPARAMETER STIRLING NUMBERS OF THE SECOND KIND

The multiparameter Stirling numbers of the second kind $S(m;\, \underset{\sim}{n},\, \underset{\sim}{r})$ are defined by their egf

$$f_{\underset{\sim}{n}}(t;\, \underset{\sim}{r}) = \sum_{m=\underset{\sim}{r}'\underset{\sim}{n}}^{\infty} S(m;\, \underset{\sim}{n},\, \underset{\sim}{r})\frac{t^m}{m!} = \prod_{i=1}^{k} \frac{1}{n_i!}\left[e^t - \sum_{j=0}^{r_i-1} \frac{t^j}{j!}\right]^{n_i}. \tag{4.1}$$

Taking $k = 1$, $r_1 = r$ gives the generalized Stirling numbers of the second kind, $S(m, n, r)$ (Charalambides [6]; taking $k = 1$, $r = 1$ defines the simple Sterling numbers $S(m, n)$. The following properties of $S(m;\, \underset{\sim}{n},\, \underset{\sim}{r})$ can easily be established (cf. §2).

a) They have the representation

$$S(m;\, \underset{\sim}{n},\, \underset{\sim}{r}) = \frac{m!}{n_1! \,\ldots\, n_k!}\sum_{m} \prod_{i=1}^{k} \prod_{j=1}^{n_i} \frac{1}{x_{ij}!}, \tag{4.2}$$

where the summation extends over all ordered N-tuples ($N = n_1 + \cdots + n_k$) of integers x_{ij} satisfying

$$x_{ij} \geqslant r_i,\ i = 1, \ldots, k \quad \text{and} \quad \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} = m.$$

b) They satisfy the following recurrence relations,

$$S(m + 1;\, \underset{\sim}{n},\, \underset{\sim}{r}) = NS(m;\, \underset{\sim}{n},\, \underset{\sim}{r}) + \sum_{i=1}^{k} \binom{m}{r_i - 1} S(m - r_i + 1;\, \underset{\sim}{n} - \underset{\sim}{e}_i,\, \underset{\sim}{r}) \tag{4.3}$$

and

$$S(m;\, \underset{\sim}{n},\, \underset{\sim}{r} + \underset{\sim}{e}_i) = \sum_{j=0}^{n_i} (-1)^j \frac{(m)_{jr_i}}{j!\,(r_i!)^j}\, S(m - jr_i;\, \underset{\sim}{n} - j\underset{\sim}{e}_i,\, \underset{\sim}{r}), \tag{4.4}$$

with initial conditions

$$S(0;\, 0,\, \underset{\sim}{r}) = 1,\ S(0;\, \underset{\sim}{n},\, \underset{\sim}{r}) = 0 \text{ whenever } \sum r_i n_i > 0$$

and

$$S(m;\, \underset{\sim}{n},\, \underset{\sim}{r}) = 0 \text{ if } m < \underset{\sim}{r}'\underset{\sim}{n}. \tag{4.5}$$

These follow from the difference/differential equation

$$\frac{d}{dt} f_{\underset{\sim}{n}}(t;\, \underset{\sim}{r}) = Nf_{\underset{\sim}{n}}(t,\, \underset{\sim}{r}) + \sum_{i=1}^{k} f_{\underset{\sim}{n}-\underset{\sim}{e}_i}(t;\, \underset{\sim}{r})t^{r_i - 1}/(r_i - 1)! \tag{4.6}$$

It can easily be seen that the representation (4.2) provides the following combinatorial interpretation in terms of occupancy numbers.

Proposition 4.1: The number of ways of placing m distinguishable balls into $N = n_1 + \cdots + n_k$ cells so that each cell of the i^{th} group of n_i cells contains at least r_i balls for $i = 1, \ldots, k$ is equal to $n_1! \ldots n_k! S(m;\, \underset{\sim}{n},\, \underset{\sim}{r})$ if the N cells are distinguishable, and is equal to $S(m;\, \underset{\sim}{n},\, \underset{\sim}{r})$ if only cells belonging to different groups are distinguishable (and cells in the same group are alike).

If is easily concluded from Proposition 4.1, or from (4.3)-(4.5), that the numbers $S(m;\, \underset{\sim}{n},\, \underset{\sim}{r})$ are nonnegative integers.

5. RATIOS OF MULTIPARAMETER STIRLING NUMBERS OF THE SECOND KIND

We define, as a ratio of multiparameter Stirling numbers of the second kind with respect to argument m, the function

$$R_2(m; \underset{\sim}{n}, \underset{\sim}{r}) = \frac{S(m+1; \underset{\sim}{n}, \underset{\sim}{r})}{S(m; \underset{\sim}{n}, \underset{\sim}{r})}. \tag{5.1}$$

Working as for Proposition 3.1, we obtain

Proposition 5.1: A recurrence relation for the ratio $R_2(m; \underset{\sim}{n}, \underset{\sim}{r})$, independent of the multiparameter Stirling numbers of the second kind, is given by

$$R_2(m; \underset{\sim}{n}, \underset{\sim}{r}) - N = \frac{\displaystyle\sum_{j=1}^{k} \frac{\binom{m}{r_j - 1} r_j!\, n_j}{(\underset{\sim}{r}'\underset{\sim}{n})_{r_j}} \prod_{i=1}^{m+1-\underset{\sim}{r}'\underset{\sim}{n}} R_2(m - r_j + 1 - i, \underset{\sim}{n} - \underset{\sim}{e}_j, \underset{\sim}{r})}{\displaystyle\prod_{i=1}^{m-\underset{\sim}{r}'\underset{\sim}{n}} R_2(m - i, \underset{\sim}{n}, \underset{\sim}{r})}, \tag{5.2}$$

for $\underset{\sim}{n} > \underset{\sim}{1}$ and $m > \underset{\sim}{r}'\underset{\sim}{n}$, with

$$R_2(m, 1, r) = k \tag{5.3}$$

and

$$R_2(\underset{\sim}{r}'\underset{\sim}{n}; \underset{\sim}{n}, \underset{\sim}{r}) = (\underset{\sim}{r}'\underset{\sim}{n} + 1)\sum_{j=1}^{k} \frac{n_j}{(r_j + 1)}. \tag{5.4}$$

The special case $k = 1$ yields

Proposition 5.2: A recurrence relation for the ratio $R_2(m, n, r)$, independent of the generalized Stirling numbers of the second kind, is given by

$$R_2(m, n, r) - n = \frac{\displaystyle\frac{n\binom{m}{r-1} r!}{(rn)_r} \prod_{i=1}^{m+1-rn} R_2(m - r + 1 - i, n - 1, r)}{\displaystyle\prod_{i=1}^{m-rn} R_2(m - i, n, r)} \tag{5.5}$$

for $n > 1$ and $m > rn$, with

$$R_2(m, 1, r) = 1 \tag{5.6}$$

and

$$R_2(rn, n, r) = n(rn + 1)/(r + 1). \tag{5.7}$$

Also for $k = 1$, $r = 1$ we obtain

Proposition 5.3: A recurrence relation for the ratio $R_2(m, n)$, independent of the usual Stirling numbers of the second kind, is given by

$$R_2(m, n) - n = \frac{\displaystyle\prod_{i=1}^{m+1-n} R_2(m - i, n - 1)}{\displaystyle\prod_{i=1}^{m-n} R_2(m - i, n)}, \tag{5.8}$$

for $n > 1$ and $m > n$, with

$$R_2(m, 1) = 1 \tag{5.9}$$

and

$$R_2(n, n) = n(n+1)/2. \tag{5.10}$$

Proposition 5.4: An alternative recurrence relation for the ratio $R_2(m, n, r)$ is given by

$$R_2(m, n, r) - n = \frac{m}{m - r + 1} \frac{[R_2(m-1, n, r) - n]R_2(m-r, n-1, r)}{R_2(m-1, n, r)}, \tag{5.11}$$

for $n > 1$ and $m > rn$.

Applying Proposition 5.4 with $r = 1$ gives

Proposition 5.5: An alternative recurrence relation for the ratio $R_2(m, n)$, is given by

$$R_2(m, n) - n = \frac{[R_2(m-1, n) - n]R_2(m-1, n-1)}{R_2(m-1, n)}, \tag{5.12}$$

for $n > 1$ and $m > n$.

The last relation was also derived by Berg [1].

6. MULTIPARAMETER C-NUMBERS

The multiparameter C-numbers, $C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r})$, are defined by their egf

$$\varphi_{\underset{\sim}{n}}(t; \underset{\sim}{s}, \underset{\sim}{r}) = \sum_{m=\underset{\sim}{r}'\underset{\sim}{n}}^{\infty} C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r})\frac{t^m}{m!} = \prod_{i=1}^{k} \frac{1}{n_i!}\left[(1+t)^{s_i} - \sum_{j=0}^{r_i-1} \binom{s_i}{j} t^j\right]^{n_i}, \tag{6.1}$$

where the $s_i \neq 0$, $i = 1, \ldots, k$, are any real numbers.

Taking $k = 1$ gives the generalized C-numbers (Charalambides [6]) and $k = 1$, $r_1 = 1$ defines the simple C-numbers (Cacoullos and Charalambides [5], Charalambides [8].

The following properties of $C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r})$ are easily verified.

a) They have the representation

$$C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r}) = \frac{m!}{n_1! \ldots n_k!} \sum_m \prod_{i=1}^{k} \prod_{j=1}^{n_i} \binom{s_i}{x_{ij}}, \tag{6.2}$$

where the summation extends over all ordered N-tuples $(N = n_1 + \cdots + n_k)$ of integers x_{ij} satisfying

$$x_{ij} \geqslant r_i, \; i = 1, \ldots, k \quad \text{and} \quad \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} = m.$$

b) They satisfy the following recurrence relations,

$$\begin{aligned} C(m+1; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r}) &= (\underset{\sim}{s}'\underset{\sim}{n} - m)C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r}) \\ &+ \sum_{i=1}^{k} \binom{m}{r_i - 1}(s_i)_{r_i} C(m - r_i + 1; \underset{\sim}{n} - \underset{\sim}{e}_i, \underset{\sim}{s}, \underset{\sim}{r}) \end{aligned} \tag{6.3}$$

and

$$C(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r} + \underset{\sim}{e}_i) = \sum_{j=0}^{n_i} (-1)^j \frac{(m)_{jr_i}}{j!}\binom{s_i}{r_i}^j C(m - jr_i;\ \underset{\sim}{n} - j\underset{\sim}{e}_i,\ \underset{\sim}{s},\ \underset{\sim}{r}), \tag{6.4}$$

with initial conditions

$$C(0;\ 0,\ \underset{\sim}{s},\ \underset{\sim}{r}) = 1,\ C(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r}) = 0 \text{ when } m < \underset{\sim}{r}'\underset{\sim}{n}.$$

They are obtained from the difference/differential equation

$$(1+t)\frac{d}{dt}\varphi_{\underset{\sim}{n}}(t;\ \underset{\sim}{s},\ \underset{\sim}{r}) = \underset{\sim}{s}'\underset{\sim}{n}\varphi_{\underset{\sim}{n}}(t;\ \underset{\sim}{s},\ \underset{\sim}{r}) + \sum_{i=1}^{k} \frac{(s_i)_{r_i}}{(r_i - 1)!} t^{r_i - 1}\varphi_{\underset{\sim}{n} - \underset{\sim}{e}_i}(t;\ \underset{\sim}{s},\ \underset{\sim}{r}). \tag{6.5}$$

The representation (6.2) leads us to the following interpretation of the $C(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r})$-numbers in the framework of coupon-collecting problems.

Consider an urn containing k groups (sets) of distinguishable balls; the i^{th} group consists of $s_i n_i$ balls and is divided into equal subgroups (subsets) of s_i balls each bearing the numbers $1, \ldots, n_i$; moreover, suppose the balls of the k groups are distinguished by different colors so that each ball in the urn is distinguished by its color and number. Now it is easily seen from (6.2) that

Proposition 6.1: The number of ways of selecting m balls out of an urn with

$$\underset{\sim}{s}'\underset{\sim}{n} = \sum_{i=1}^{k} s_i n_i$$

distinguishable balls, divided into k groups by color and number as above into n_i subsets of size s_i within the i^{th} subgroup, so that each number $1, \ldots, n_i$ of the i^{th} subgroup (color) appears at least r_i times is equal to

$$\frac{n_1! \ldots n_k!}{m!} C(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r}). \tag{6.6}$$

Here it was assumed that s_i is a positive integer. If s_i is a negative integer, say $s_i = -s_i^*$, then

$$\prod_{i=1}^{k}\prod_{j=1}^{n_i}\binom{s_i}{x_{ij}} = \prod_{i=1}^{k}\prod_{j=1}^{n_i}\binom{-s_i^*}{x_{ij}} = \prod_{i=1}^{k}\prod_{j=1}^{n_i}(-1)^{x_{ij}}\binom{s_i^* + x_{ij} - 1}{x_{ij}} \tag{6.7}$$

and from (6.2) it can be concluded that the sign of $C(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r})$ is the same as $(-1)^m$. Furthermore, we may deduce

Proposition 6.2: The number of ways of distributing m $(m \geqslant \underset{\sim}{r}'\underset{\sim}{n})$ nondistinguishable balls into $\underset{\sim}{s}^{*\prime}\underset{\sim}{n}$ cells, divided into k groups of cells with $s_i n_i$ cells in the i^{th} group and n_i subgroups each of s_i cells in the i^{th} group, so that each subgroup of the i^{th} group contains at least r_i balls is equal to

$$\frac{n_1! \ldots n_k!}{m!}\left|C(m;\ \underset{\sim}{n},\ -\underset{\sim}{s}^*,\ \underset{\sim}{r})\right|. \tag{6.8}$$

As an indication of the applicability of the multiparameter C-numbers in occupancy problems, we refer to a problem posed by Sobel *et al.* [12, p. 52].

Signless Multiparameter C-Numbers

From the basic recurrence relation (6.3) or from the last two propositions, we conclude that:

a) for $s_i > 0$ an integer, the numbers $C(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r})$ are nonnegative integers; they are positive for $\underset{\sim}{r}'\underset{\sim}{n} \leqslant m \leqslant \underset{\sim}{s}'\underset{\sim}{n}$; otherwise zero;

b) for $s_i < 0$ an integer, the numbers $C(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r})$ are integers having the sign of $(-1)^m$.

Thus, as in the case of the Stirling numbers of the first kind, Riordan [10], the positive numbers

$$|C(m;\ \underset{\sim}{n},\ -\underset{\sim}{s}^*,\ \underset{\sim}{r})| = (-1)^m C(m;\ \underset{\sim}{n},\ -\underset{\sim}{s}^*,\ \underset{\sim}{r}) \tag{6.9}$$

will be called signless multiparameter C-numbers.

It can easily be verified that

Proposition 6.3: The egf of the signless multiparameter C-numbers

$$|C(m;\ \underset{\sim}{n},\ -\underset{\sim}{s},\ \underset{\sim}{r})|,$$

$s_i > 0$, $i = 1, \ldots, k$, is given by

$$\varphi_n^*(t;\ -\underset{\sim}{s},\ \underset{\sim}{r}) = \prod_{i=1}^{k} \frac{1}{n_i!}\left[(1 - t)^{-s_i} - \sum_{j=0}^{r_i - 1} (-1)^j \binom{-s_i}{j} t^j\right]^{n_i}. \tag{6.10}$$

Remark: It should be observed that this is exactly the egf required for the treatment of the mvue problem in the negative binomial case when the probability function of the i^{th} sample is

$$\begin{aligned} P(X = x_{ij}) &= \frac{1}{g(\theta,\ r_i)}\binom{s_i + x_{ij} - 1}{x_{ij}}\theta^{x_{ij}}(1 - \theta)^{s_i} \\ &= (-1)^{x_{ij}}\binom{-s_i}{x_{ij}}\theta^{x_{ij}}(1 - \theta)^{s_i} \end{aligned} \tag{6.11}$$

with

$$g(\theta,\ r_i) = (1 - \theta)^{-s_i} - \sum_{j=0}^{r_i - 1} (-1)^j \binom{-s_i}{j}\theta^j,\ i = 1, \ldots, k. \tag{6.12}$$

7. RATIOS OF MULTIPARAMETER C-NUMBERS

We define, as a ratio of multiparameter C-numbers with respect to argument m, the function

$$R_3(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r}) = \frac{C(m + 1;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r})}{C(m;\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r})} \tag{7.1}$$

Proposition 7.1: A recurrence relation for the ratio $R_3(m,\ \underset{\sim}{n},\ \underset{\sim}{s},\ \underset{\sim}{r})$, independent of the multiparameter C-numbers, is given by

$$R_3(m;\, \underset{\sim}{n},\, \underset{\sim}{s},\, \underset{\sim}{r}) + m - \underset{\sim}{s}'\underset{\sim}{n} = \frac{\displaystyle\sum_{j=1}^{k} \frac{\binom{m}{r_j - 1}(s_j)_{r_j}\, n_j}{(\underset{\sim}{r}'\underset{\sim}{n})_{r_j}\binom{s_j}{r_j}} \prod_{i=1}^{m+1-\underset{\sim}{r}'\underset{\sim}{n}} R_3(m - r_j + 1 - i,\, \underset{\sim}{n} - \underset{\sim}{e}_j,\, \underset{\sim}{s},\, \underset{\sim}{r})}{\displaystyle\prod_{i=1}^{m-\underset{\sim}{r}'\underset{\sim}{n}} R_3(m - i,\, \underset{\sim}{n},\, \underset{\sim}{s},\, \underset{\sim}{r})} \tag{7.2}$$

for $\underset{\sim}{n} > \underset{\sim}{1}$ and $m > \underset{\sim}{r}'\underset{\sim}{n}$, with

$$R_3(m,\, 1,\, s,\, r) = s - m \tag{7.3}$$

and

$$R_3(\underset{\sim}{r}'\underset{\sim}{n};\, \underset{\sim}{n},\, \underset{\sim}{s},\, \underset{\sim}{r}) = (\underset{\sim}{r}'\underset{\sim}{n} + 1)\sum_{j=1}^{k} \frac{n_j(s_j - r_j)}{(r_j + 1)}. \tag{7.4}$$

Proposition 7.2: A recurrence relation for the ratio $R_3(m,\, n,\, s,\, r)$, independent of the generalized C-numbers (case $k = 1$), is given by

$$R_3(m,\, n,\, s,\, r) + m - sn = \frac{\displaystyle\frac{\binom{m}{r - 1}(s)_r\, n}{(rn)_r\binom{s}{r}} \prod_{i=1}^{m+1-rn} R_3(m - r + 1 - i,\, n - 1,\, s,\, r)}{\displaystyle\prod_{i=1}^{m-rn} R_3(m - i,\, n,\, s,\, r)}, \tag{7.5}$$

for $n > 1$ and $m > rn$, with

$$R_3(m,\, 1,\, s,\, r) = s - m \tag{7.6}$$

and

$$R_3(rn,\, n,\, s,\, r) = n(rn + 1)(s - r)/(r + 1). \tag{7.7}$$

Proposition 7.3: A recurrence relation for the ratio $R_3(m,\, n,\, s)$, independent of the usual C-numbers (case $r = 1$), is given by

$$R_3(m,\, n,\, s) + m - sn = \frac{\displaystyle\prod_{i=1}^{m+1-n} R_3(m - i,\, n - 1,\, s)}{\displaystyle\prod_{i=1}^{m-n} R_3(m - i,\, n,\, s)}, \tag{7.8}$$

for $n > 1$ and $m > n$, with

$$R_3(m,\, 1,\, s) = s - m \tag{7.9}$$

and

$$R_3(n,\, n,\, s) = (s - 1)n(n + 1)/2. \tag{7.10}$$

Proposition 7.4: An alternative recurrence relation for the ratio $R_3(m,\, n,\, s,\, r)$ is given by

$$R_3(m,\, n,\, s,\, r) + m - sn = \frac{m}{m - r + 1}\, \frac{[R_3(m - 1,\, n,\, s,\, r) + m - sn - 1]R_3(m - r,\, n - 1,\, s,\, r)}{R_3(m - 1,\, n,\, s,\, r)}, \tag{7.11}$$

for $n > 1$ and $m > rn$.

Proposition 7.5: An alternative recurrence relation for the ratio $R_3(m, n, s)$ is given by

$$R_3(m, n, s) + m - sn = \frac{[R_3(m-1, n, s) + m - sn - 1]R_3(m-1, n-1, s)}{R_3(m-1, n, s)}, \qquad (7.12)$$

for $n > 1$ and $m > n$.

8. RELATIONS BETWEEN THE STIRLING AND C-NUMBERS

It was observed in Cacoullos and Charalambides [5] that

$$\lim_{s_i \to \pm\infty} s^{-m}C(m, n, s) = S(m, n); \qquad (8.1)$$

i.e., the C-numbers can be approximated by the Stirling numbers of the second kind for large s, a fact that reflects the corresponding well-known convergence of the binomial to the Poisson ($s \to \infty$, $p \to 0$, i.e., $\theta = p/q \to 0$ and, hence, sp or $s\theta$ converges to the Poisson parameter λ). The above property extends to the case of multiparameter Stirling numbers of the second kind and multiparameter C-numbers; namely,

$$\lim_{s_i \to \pm\infty} s_i^{-m}C(m; \underset{\sim}{n}, \underset{\sim}{s}, \underset{\sim}{r}) = S(m; \underset{\sim}{n}, \underset{\sim}{r}), \; i = 1, \ldots, k. \qquad (8.2)$$

This can be verified by using the corresponding representations (4.2) and (6.2) of these numbers and noting that

$$\lim_{s_i \to \pm\infty} s^{-k}\binom{s}{k} = 1/k!. \qquad (8.3)$$

A relationship between the signless multiparameter Stirling numbers of the first kind and the multiparameter C-numbers reflects the limiting relationship between the negative binomial and the logarithmic series distributions:

$$\lim_{s_i \to 0} s_i^{-N}|C(m; \underset{\sim}{n}, -\underset{\sim}{s}, \underset{\sim}{r})| = |s(m; \underset{\sim}{n}, \underset{\sim}{r})|, \; N = n_1 + \cdots + n_k. \qquad (8.4)$$

This can be seen, e.g., by showing that the egf of the $s_i^{-N}|C(m, \underset{\sim}{n}, -\underset{\sim}{s}, \underset{\sim}{r})|$-numbers converge to the egf of the $|s(m; \underset{\sim}{n}, \underset{\sim}{r})|$-numbers; i.e.,

$$\begin{aligned}\lim_{s_i \to 0} \frac{1}{s_i^N}\prod_{i=1}^{k}\frac{1}{n_i!}&\left[(1-t)^{-s_i} - \sum_{j=0}^{r_i-1}(-1)^j\binom{-s_i}{j}t^j\right]^{n_i} \\ &= \prod_{i=1}^{k}\frac{1}{n_i!}\left[-\log(1-t) - \sum_{j=1}^{r_i-1}\frac{t^j}{j}\right]^{n_i}. \qquad (8.5)\end{aligned}$$

For this, note that

$$\frac{1}{s}(-1)^j\binom{-s}{j}t^j \xrightarrow[s \to 0]{} \frac{t^j}{j}. \qquad (8.6)$$

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