ON THE ASYMPTOTIC PROPORTIONS OF ZEROS AND ONES IN FIBONACCI SEQUENCES

PETER H. ST. JOHN
Computer Sciences Corp., Piscataway, NJ 08854
(Submitted July 1982)

By "Fibonacci sequence" we mean a binary sequence such that no two one's, say, are consecutive, with unrestricted first entry; hence, the number of such sequences of length n is f_{n+1}.

It is understood that

$$f_n = c\alpha^n + \overline{c}\overline{\alpha}^n \text{ with } \alpha = \frac{1+\sqrt{5}}{2} \text{ (the "golden ratio")}, \tag{1}$$

$$c = \frac{5+\sqrt{5}}{10},\ \overline{\alpha} = 1 - \alpha \text{ and } \overline{c} = 1 - c.$$

We denote by p and q the asymptotic proportions of zeros and ones, respectively, in Fibonacci sequences, so that $p + q = 1$. We will show

Theorem:

$$p = c \text{ and } q = \overline{c}. \tag{2}$$

Let ω_n be the total number of ones in all Fibonacci sequences of length n; hence, $\omega_0 = 0$ and $\omega_1 = 1$. Since the total number of ones in all n-sequences is the number in all $(n - 1)$-sequences, with zeros appended to the ends, plus the number in all $(n - 2)$-sequences, with zero-ones appended, plus the number of ones in those zero-ones, we have

$$\omega_n = \omega_{n-1} + \omega_{n-2} + f_{n-1}. \tag{3}$$

We know that such a recursion [1, p. 101] gives

$$\omega_{n+1} = \sum_{k=0}^{n} f_k f_{n-k}. \tag{4}$$

The proportion of ones is the number of ones divided by the number of entries—n per sequence times f_{n+1} sequences—so we define

$$q_n = \frac{\omega_n}{nf_{n+1}} \quad \text{and} \quad q = \lim_{n\to\infty} q_n. \tag{5}$$

Clearly, the limit exists and is less than 1/2, as the ones are restricted but the zeros are not.

From (1) and (4), we have

$$\omega_{n+1} = \sum_{k=0}^{n} (c\alpha^k + \overline{c}\overline{\alpha}^k)(c\alpha^{n-k} + \overline{c}\overline{\alpha}^{n-k}) \tag{6a}$$

$$\omega_{n+1} = (n+1)(c^2\alpha^n + \overline{c}^2\overline{\alpha}^n) + c\overline{c}\sum_{k=0}^{n}(\alpha^k\overline{\alpha}^{k-n} + \alpha^{n-k}\overline{\alpha}^k). \tag{6b}$$

As $\alpha\overline{\alpha} = -1$, the indexed sum on the right of (6b) is

$$c\bar{c}\sum_{k=0}^{n}[(-1)^k\bar{\alpha}^{n-2k} + (-1)^k\alpha^{n-2k}] \tag{7a}$$

and by inverting the order of summation on the left,

$$2c\bar{c}\sum_{k=0}^{n}(-1)^{n-k}\alpha^{n-2k} \tag{7b}$$

which is clearly less than

$$\frac{\alpha^{2n+2} - 1}{\alpha^n(\alpha^2 - 1)}\, 2c\bar{c} = o(n\alpha^n) \tag{7c}$$

where $f(n) = o(g(n))$ means $\lim\limits_{n\to\infty}\dfrac{f(n)}{g(n)} = 0$.

Substituting (7c) into (6b), and thence into (5), we have

$$q_{n+1} = \frac{c^2\alpha^n + \bar{c}^2\bar{\alpha}^n}{c\alpha^{n+2} + \bar{c}\bar{\alpha}^{n+2}} + o(1); \tag{8}$$

as $\alpha > 1$ and $|\bar{\alpha}| < 1$, and taking $n \to \infty$,

$$q = \frac{c}{\alpha^2} = \frac{5 + \sqrt{5}}{10} \cdot \frac{2}{3 + \sqrt{5}} = \frac{5 - \sqrt{5}}{10} = \bar{c}, \tag{9}$$

and hence the theorem.

ACKNOWLEDGMENT

The author wishes to thank Mr. Hugh Bender for providing helpful numerical evidence.

REFERENCE

1. John Riordan. *Combinatorial Identities*. New York: Wiley, 1968.

◆◇◆◇◆