ON CERTAIN SERIES OF RECIPROCALS OF FIBONACCI NUMBERS

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The purpose of this note is to give an alternative, shorter proof of a result of R. P. Backstrom concerning the sums of series whose terms are reciprocals of Fibonacci numbers, a problem on which much interest has recently been focused.

Furthermore, the method used here gives the possibility of obtaining new formulas related to the Fibonacci and Lucas numbers.

In fact, we establish in explicit form series of the form

$$\sum_{n=0}^{\infty} \frac{1}{F_{an+b} \pm c}, \quad \sum_{n=0}^{\infty} \frac{1}{F_{an+b}F_{cn+d}}, \quad \sum_{n=0}^{\infty} \frac{1}{F_{an+b}^2 \pm F_{cn+d}^2},$$

for certain values of a, b, c, and d.

We start with the identity

$$F_n - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2, \tag{1}$$

which, by replacing n with $(2n+1)r + 2k$, becomes

$$F_{2(n+1)r+2k}^2 - F_r^2 = F_{2nr+2k}\,F_{2(n+1)r+2k}. \tag{2}$$

Then

$$\frac{1}{F_{2(n+1)r+2k} + F_r} = \frac{F_{(2n+1)r+2k} - F_r}{F_{2nr+2k}\,F_{2(n+1)r+2k}} \tag{3}$$

with $-(r-1) \leqslant 2k \leqslant r-1$.

Since

$$L_rF_{2(nr+k)+r} = F_{2(nr+k)+2r} + (-1)^rF_{2(nr+k)},$$

from (3) we obtain

$$\frac{1}{F_{(2n+1)r+2k} + F_r} = \frac{1}{L_r}\left(\frac{1}{F_{2nr+2k}} + \frac{(-1)^r}{F_{2(n+1)r+2k}}\right) - \frac{F_r}{F_{2nr+2k}\,F_{2(n+1)r+2k}}.$$

Now, consider the sum

$$\begin{aligned} S_N(r, k) &= \sum_{n=0}^{N} \frac{1}{F_{(2n+1)r+2k} + F_r} \\ &= \frac{1}{L_r}\sum_{n=0}^{N}\left(\frac{1}{F_{2nr+2k}} + \frac{(-1)^r}{F_{2(n+1)r+2k}}\right) - F_r\sum_{n=0}^{N} \frac{1}{F_{2nr+2k}\,F_{2(n+1)r+2k}}. \end{aligned}$$

We have

$$\sum_{n=0}^{N}\left(\frac{1}{F_{2nr+2k}} + \frac{(-1)^r}{F_{2(n+1)r+2k}}\right) = \frac{1}{F_{2k}} - \frac{1}{F_{2(N+1)r+2k}},$$

for an odd integer r, and

$$\sum_{n=0}^{N} \frac{1}{F_{2nr+2k}\,F_{2(n+1)r+2k}} = \frac{1}{2F_{2r}}\left(\frac{L_{2k}}{F_{2k}} - \frac{L_{2(N+1)r+2k}}{F_{2(N+1)r+2k}}\right), \tag{4}$$

which follows from the identity

$$\frac{L_{2k}}{F_{2k}} - \frac{L_{2k+2r}}{F_{2k+2r}} = \frac{2F_{2r}}{F_{2k}F_{2k+2r}},$$

if we successively replace k by k, $k+r$, ..., $k+Nr$, and sum the obtained equations.

Therefore,

$$S_N(r,\ k) = \frac{1}{2L_r}\left(\frac{2 - L_{2k}}{F_{2k}} + \frac{L_{2(N+1)r+2k} - 2}{F_{2(N+1)r+2k}}\right).$$

Using the relations

$$L_{2n} = L_n^2 - 2(-1)^n = 5F_n^2 + 2(-1)^n,$$

it follows that (for odd integer r)

$$S_N(r,\ k) = \sum_{n=0}^{N} \frac{1}{F_{(2n+1)r+2k} + F_r} = \begin{cases} \left(\dfrac{L_{(N+1)r+k}}{F_{(N+1)r+k}} - \dfrac{5F_k}{L_k}\right)\Big/ 2L_r, & N\text{-even, } k\text{-even,} \\ \left(\dfrac{5F_{(N+1)r+k}}{L_{(N+1)r+k}} - \dfrac{L_k}{F_k}\right)\Big/ 2L_r, & N\text{-even, } k\text{-odd,} \\ \left(\dfrac{5F_{(N+1)r+k}}{L_{(N+1)r+k}} - \dfrac{5F_k}{L_k}\right)\Big/ 2L_r, & N\text{-odd, } k\text{-even,} \\ \left(\dfrac{L_{(N+1)r+k}}{F_{(N+1)r+k}} - \dfrac{L_k}{F_k}\right)\Big/ 2L_r, & N\text{-odd, } k\text{-odd.} \end{cases} \tag{5}$$

Letting $N \to \infty$, we have

$$S(r,\ k) = \sum_{n=0}^{\infty} \frac{1}{F_{(2n+1)r+2k} + F_r} = \begin{cases} \dfrac{1}{2L_r}\left(\sqrt{5} - \dfrac{5F_k}{L_k}\right), & k\text{-even,} \\ \dfrac{1}{2L_r}\left(\sqrt{5} - \dfrac{L_k}{F_k}\right), & k\text{-odd.} \end{cases} \tag{5a}$$

Summing $S(r,\ k)$ over the r values of k finally yields

$$S(r) = \sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + F_r} = \frac{r\sqrt{5}}{2L_r},$$

by using the relations $F_{-n} = (-1)^n F_n$ and $L_{-n} = (-1)^n L_n$.

Following arguments similar to the above for obtaining (5), we have

$$\overline{S}_N(r,\ k) = \sum_{n=0}^{N} \frac{1}{F_{(2n+1)r+2k} - F_r} \tag{6}$$

$$= \frac{1}{2L}\left(\frac{2 + L_{2k}}{F_{2k}} - \frac{L_{2(N+1)r+2k} + 2}{F_{2(N+1)r+2k}}\right)$$

(continued)

$$= \begin{cases} \left(\dfrac{5F_k}{L_k} - \dfrac{L_{(N+1)r+k}}{F_{(N+1)r+k}}\right)\Big/ 2L_r, & N\text{-even}, \ k\text{-odd}, \\ \left(\dfrac{L_k}{F_k} - \dfrac{5F_{(N+1)r+k}}{L_{(N+1)r+k}}\right)\Big/ 2L_r, & N\text{-even}, \ k\text{-even}, \\ \left(\dfrac{5F_k}{L_k} - \dfrac{5F_{(N+1)r+k}}{L_{(N+1)r+k}}\right)\Big/ 2L_r, & N\text{-odd}, \ k\text{-odd}, \\ \left(\dfrac{L_k}{F_k} - \dfrac{L_{(N+1)r+k}}{F_{(N+1)r+k}}\right)\Big/ 2L_r, & N\text{-odd}, \ k\text{-even}. \end{cases}$$

whenever $k \geqslant 1$ and r is odd.

Comparing (5) and (6) by letting k be even, $k = 2s$ in (6), and k be odd, $k = 2t - 1$ in (5), we see that if r is odd, then

$$\overline{S}_N(r, 2s) = -S_N(r, 2t - 1).$$

Similarly, with $k = 2s - 1$ in (6) and $k = 2t$ in (5), we have, for r odd, that

$$\overline{S}_N(r, 2s - 1) = -S_N(r, 2t).$$

Letting $N \to \infty$ in (6), we have, for odd r, that

$$\overline{S}(r, k) = \sum_{n=0}^{\infty} \frac{1}{F_{2(n+1)r+2k} - F_r} = \begin{cases} \left(\dfrac{5F_k}{L_k} - \sqrt{5}\right)\Big/ 2L_r, & k\text{-odd}, \\ \left(\dfrac{L_k}{F_k} - \sqrt{5}\right)\Big/ 2L_r, & k\text{-even}. \end{cases} \tag{6a}$$

Comparing (5a) and (6a) as above, we see that if $k = 2s - 1$ in (6a) and $k = 2t$ in (5a), then for r odd,

$$\overline{S}(r, 2s - 1) = -S(r, 2t),$$

while $k = 2s$ in (6a) and $k = 2t - 1$ in (5a) yields

$$\overline{S}(r, 2s) = -S(r, 2t - 1), \text{ if } r \text{ is odd.}$$

We note that, from (1), it follows that

$$\frac{1}{F_{2(n+1)+r} - F_r} - \frac{1}{F_{2(n+1)+r} + F_r} = \frac{2F_r}{F_{2(n+1)+2r}F_{2(n+1)}}.$$

Hence, we have

$$\sum_{n=0}^{N} \frac{1}{F_{2(n+1)+r} - F_r} = \sum_{n=0}^{N} \frac{1}{F_{2(n+1)+r} + F_r} + 2F_r \sum_{n=0}^{N} \frac{1}{F_{2(n+1)}F_{2(n+1)+2r}}.$$

Taking (4) into consideration, along with the fact that $\lim\limits_{n\to\infty} \dfrac{L_n}{F_n} = \sqrt{5}$, we obtain

$$\sum_{n=0}^{\infty} \frac{1}{F_{2nr+2k}F_{2(n+1)r+2k}} = \frac{1}{2F_{2r}}\left(\frac{L_{2k}}{F_{2k}} - \sqrt{5}\right), \text{ if } r \text{ is odd.}$$

Similarly, for the Lucas numbers, starting from

$$L_n^2 - 5F_{n-r}F_{n+r} = (-1)^{n-r}L_r^2,$$

we find that

$$G\ (r,\ k) = \sum_{n=0}^{N} \frac{1}{L_{(2n+1)r+2k} + L_r} = \frac{1}{10F_r}\left(\frac{2 - L_{2k}}{F_{2k}} + \frac{L_{2(N+1)r+2k} - 2}{F_{2(N+1)r+2k}}\right),$$

with $-r \leqslant 2k \leqslant r - 2$ and r an even integer.

Following the methods used above, we obtain, for r even,

$$G_N(r,\ k) = \begin{cases} \dfrac{1}{10F_r}\left(\dfrac{L_{(N+1)r+k}}{F_{(N+1)r+k}} - \dfrac{L_k}{F_k}\right), & k\text{-odd}, \\[2ex] \dfrac{1}{10F_r}\left(\dfrac{5F_{(N+1)r+k}}{L_{(N+1)r+k}} - \dfrac{5F_k}{L_k}\right), & k\text{-even}. \end{cases} \tag{7}$$

Letting $N \to \infty$, this relation yeilds, for r even,

$$G(r,\ k) = \sum_{n=0}^{\infty} \frac{1}{L_{(2n+1)r+2k} + L_r} = \begin{cases} \dfrac{1}{10F_r}\left(\sqrt{5} - \dfrac{L_k}{F_k}\right), & k\text{-odd}, \\[2ex] \dfrac{1}{10F_r}\left(\sqrt{5} - \dfrac{5F_k}{L_k}\right), & k\text{-even}. \end{cases} \tag{7a}$$

Summing the last equation over the $r - 1$ values of k, leads to

$$G(r) = \sum_{n=0}^{\infty} \frac{1}{L_{2n} + L_r} = \begin{cases} \dfrac{r\sqrt{5}}{10F_r} + \dfrac{1}{10F_{r/2}^2}, & r/2\text{-odd} \\[2ex] \dfrac{r\sqrt{5}}{10F_r} + \dfrac{1}{2L_{r/2}^2}, & r/2\text{-even}. \end{cases}$$

when r is even.

Similarly, for r even,

$$\overline{G}_N(r,\ k) = \sum_{n=0}^{N} \frac{1}{L_{(2n+1)r+2k} - L} = \begin{cases} \dfrac{1}{10F_r}\left(\dfrac{L_k}{F_k} - \dfrac{L_{(N+1)r+k}}{F_{(N+1)r+k}}\right), & k\text{-even}, \\[2ex] \dfrac{1}{10F_r}\left(\dfrac{5F_k}{L_k} - \dfrac{5F_{(N+1)r+k}}{L_{(N+1)r+k}}\right), & k\text{-odd}. \end{cases} \tag{8}$$

so that

$$\overline{G}(r,\ k) = \sum_{n=0}^{\infty} \frac{1}{L_{2(n+1)r+2k} - L_r} = \begin{cases} \dfrac{1}{10F_r}\left(\dfrac{5F_k}{L_k} - \sqrt{5}\right), & k\text{-odd}, \\[2ex] \dfrac{1}{10F_r}\left(\dfrac{L_k}{F_k} - \sqrt{5}\right), & k\text{-even}. \end{cases} \tag{8a}$$

Comparing (7a) and (8a) as we did (5a) and (6a), we have, for r even, $r = 2s - 1$ in (8a) and $k = 2t$ in (7a), that

$$G(r,\ 2s - 1) = -\overline{G}(r,\ 2t)$$

while, for r even, $k = 2s$ in (8a) and $k = 2t - 1$ in (7a), we have

$$\overline{G}(r, 2s) = -G(r, 2t - 1).$$

By similar methods, the relations (1) and (4) can also be used to show that

$$\sum_{n=0}^{N} \frac{1}{F_{(2n+1)r+k}^2 - (-1)^k F_r^2} = \frac{F_{2(N+1)r}}{F_k F_{2r} F_{2(N+1)r+k}}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{F_{(2n+1)r+k}^2 - (-1)^k F_r^2} = \frac{(\sqrt{5} - 1)^k}{2^k F_k F_{2r}}.$$

REFERNECE

1. R. P. Backstrom. "On Reciprocal Series Related to Fibonacci Numbers with Subscripts in Arithmetic Progression." *The Fibonacci Quarterly* 19 (1981): 14-21.

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