ADVANCED PROBLEMS AND SOLUTIONS

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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-376 *Proposed by H. Klauser, Zurich, Switzerland*

Let (a, b, c, d) be a quadruple of integers with the property that

$$(a^3 + b^3 + c^3 + d^3) = 0.$$

Clearly, at least one integer must be negative.

Examples: (3, 4, 5, -6), (9, 10, -1, -12)

Find a construction rule so that:

1. out of two given quadruples a new quadruple arises;
2. out of the given quadruple a new quadruple always arises.

H-377 *Proposed by Lawrence Somer, Washington, D.C.*

Let $\{w_n\}_{n=0}^{\infty}$ be a k^{th}-order linear integral recurrence satisfying the recursion relation

$$w_{n+k} = a_1 w_{n+k-1} + a_2 w_{n+k-2} + \cdots + a_k w_n.$$

Let t be a fixed positive integer and d a fixed nonnegative integer. Show that the sequence

$$\{s_n\} = \{w_{tn+d}\}_{n=0}^{\infty}$$

also satisfies a k^{th}-order linear integral recursion relation

$$s_{n+k} = a_1^{(t)} s_{n+k-1} + a_2^{(t)} s_{n+k-2} + \cdots + a_k^{(t)} s_n.$$

Show further that the coefficients $a_1^{(t)}, a_2^{(t)}, \ldots, a_k^{(t)}$ depend on t but not on d, and that $a_k^{(t)}$ can be chosen so that

$$a_k^{(t)} = (-1)^{(k+1)(t+1)} a_k^t.$$

H-378 *Proposed by M. Wachtel, Zurich, Switzerland*

For every positive integer x and y, provided that they are *prime to each other*, show that no integral divisor of

$$x^2 - 5y^2$$

is congruent to 3 or 7, modulo 10.

H-379 *Proposed by A. N. Philippou and F. S. Makri, Univ. of Patras, Greece*

For each fixed integer $k \geqslant 2$, let $\{f_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order $k(1)$. Show that

$$f_{n+2}^{(k)} = \sum_{i=0}^{\infty} \sum_{j=0}^{n} (-1)^i \binom{n - ik}{n - j}\binom{n - j + 1}{i}, \; n \geqslant 0.$$

Reference: A. N. Phliippou & A. A. Muwafi. "Waiting for the k Consecutive Success and the Fibonacci Sequence of Order K." *The Fibonacci Quarterly* 20, no. 1 (1982):28-32.

H-380 *Proposed by Charles R. Wall, Trident Tech. College, Charleston, SC*

The sequence 1, 4, 5, 9, 13, 14, 16, 25, 29, 30, 36, 41, 49, 50, 54, 55, ... of squares and sums of consecutive squares appeared in Problem B-495. Show that this sequence has Schnirelmann density zero.

SOLUTIONS

A reply from M. Wachtel regarding H-335 (May 1983)

In the February 1984 issue, the proposer is claiming that the solution to the above-mentioned problem is incorrect.

Reply: The roots of the polynomial, as split up by the proposer, are:

$$\begin{array}{ll} (x - 1) & x_0 = 1 \\ (x^2 + bx - a^2) & x_{1,2} = \pm\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1 \\ (x^2 + ax - b^2) & x_{3,4} = \pm\sqrt{30 - 6\sqrt{5}} - (\sqrt{5} + 1) \end{array}$$

These roots are exactly identical to those shown in my solution published in the May 1983 issue, with one exception:

> As far as $x_{3,4}$ are concerned, I have erroneously omitted to apply the parentheses $-(\sqrt{5} + 1)$, sorry. Apart from this error, I do not see why this solution should be incorrect. Certainly, the solution by the proposer is more obvious.

Sum Difference!

H-355 *Proposed by Gregory Wulczyn, Bucknell Univ., Lewisburg, PA (Vol. 21, no. 2, May 1983)*

Solve the second-order finite difference equation

$$n(n - 1)a_n - \{2rn - r(r + 1)\}a_{n-r} + r^2 a_{n-2r} = 0,$$

r and n are integers. If $n - kr < 0$, $a_{n-kr} = 0$.

Solution by Paul S. Bruckman, Sacramento, CA

Let

$$y = f_r(x) = \sum_{n=0}^{\infty} a_n x^n \text{ (where the } a_n \text{ depend on } r\text{).} \tag{1}$$

We deal with four separate cases.

Case I: $r < 0$

Letting $r = -s$, the given recursion becomes

$$n(n-1)a_n + (2sn + s^{(2)})a_{n+s} + s^2 a_{n+2s} = 0,$$

or, equivalently,

$$s^2 a_n + (2sn - 3s^2 - s)a_{n-s} + (n-2s)^{(2)}a_{n-2s} = 0. \tag{2}$$

Letting $n = 0, 1, 2, \ldots$, successively, we find that (2) has only the trivial solution

$$a_n = 0,\ n = 0, 1, 2, \ldots . \tag{3}$$

Case II: $r = 0$

The given recursion becomes $n(n-1)a_n = 0$. This implies

$$a_n = 0,\ n = 2, 3, \ldots, \text{ with } a_0 \text{ and } a_1 \text{ arbitrary.} \tag{4}$$

Case III: $r = 1$

The given recursion becomes

$$n(n-1)a_n - 2(n-1)a_{n-1} + a_{n-2} = 0.$$

Again we find that a_0 and a_1 are arbitrary. Making the substitution $b_n = n!a_n$, then $b_n - 2b_{n-1} + b_{n-2} = 0$, i.e., $\Delta^2 b_n = 0$, $n = 0, 1, \ldots$. Hence, $b_n = A + Bn$ for some constants A and B. To find A and B, note

$$b_0 = A = a_0,\ b_1 = A + B = a_1,$$

so

$$A = a_0,\ B = a_1 - a_0.$$

Hence,

$$a_n = \frac{a_0 + (a_1 - a_0)n}{n!},\ n = 0, 1, 2, \ldots . \tag{5}$$

Case IV: $r \geqslant 2$

We transform the given recursion into a differential equation:

$$x^2 y'' - 2rx^{r+1}y' + (r^2 x^{2r} - r^{(2)}x^r)y = 0. \tag{6}$$

We may verify (6) by using (1) and noting that the left member of (6) becomes:

$$\sum_{n=0}^{\infty} n^{(2)}a_n x^n - 2r\sum_{n=r}^{\infty}(n-r)a_{n-r}x^n + r^2\sum_{n=2r}^{\infty} a_{n-2r}x^n - r^{(2)}\sum_{n=r}^{\infty} a_{n-r}x^n$$

$$= \sum_{n=0}^{r-1} n^{(2)}a_n x^n + \sum_{n=r}^{2r-1}\{n^{(2)}a_n - (2rn - r(r+1))a_{n-r}\}x^n$$

$$+ \sum_{n=2r}^{\infty}\{n^{(2)}a_n - (2rn - r(r+1))a_{n-r} + r^2 a_{n-2r}\}x^n = 0,$$

since each sum vanishes, using the recursion.

To solve (6), we make the fortuitous substitution $y = ue^{x^r}$, where u is some function of x. We find

$$y' = (rx^{r-1}u + u')e^{x^r}, \quad y'' = (r^2x^{2r-2}u + 2rx^{r-1}u' + r^{(2)}x^{r-2}u + u'')e^{x^r}.$$

Then, eliminating the factor e^{x^r} and simplifying, we obtain

$$x^2u'' = 0. \tag{7}$$

Since (7) is to be valid for all x, we may also eliminate the factor x^2. Then $u'' = 0$, which implies $u = A + Bx$ for some constants A and B. Thus,

$$y = f_r(x) = (A + Bx)e^{x^r}. \tag{8}$$

Since $f\ (0) = A = a_0$ and $f_r'(0) = B = a_1$, we have

$$f_r(x) = (a_0 + a_1x)e^{x^r}. \tag{9}$$

Therefore,

$$f_r(x) = a_0\sum_{n=0}^{\infty} x^{rn}/n! + a_1\sum_{n=0}^{\infty} x^{rn+1}/n!.$$

This shows that

$$a_n = \begin{cases} a_0/(n/r)!, & \text{if } r|n; \\ a_1/(n-1/r)!, & \text{if } r|(n-1); \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

An equivalent and compact formulation is the following:

$$a_n = \{\delta(r|n) + \delta(r|(n-1)) - \delta(r|n)\delta(r|(n-1))\}a_{n-rm}/m!, \tag{11}$$

where

$$m = [n/r] \text{ and } \delta(r|k) = \begin{cases} 1 & \text{if } r|k, \\ 0 & \text{otherwise.} \end{cases}$$

Lotsa Words

H-356 *Proposed by David Singmaster, Polytechnic of the South Bank, London (Vol. 21, no. 3, August 1983)*

Consider a set of r types of letter with n_i occurrences of letter i. How many words can we form, using some or all of these letters?

If we use k_i of letter i, then there are obviously

$$\binom{\Sigma k_i}{k_1, \ldots, k_r}$$

ways to form a word and the desired number is

$$\sum_{k_i \leqslant n_i}\binom{\Sigma k_i}{k_1, \ldots, k_r}.$$

When $r = 2$, this can be readily evaluated using properties of Pascal's triangle to obtain

$$\binom{n_1 + n_2 + 2}{n_1 + 1} - 1.$$

W. O. J. Moser has found a nice combinatorial derivation of this result, but neither approach works for $r > 2$.

Moser's solution for $r = 2$ follows.

In the case $r = 2$,

$$(**)\ \sum_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n}} \binom{i+j}{i}$$

is the number of ways of forming words with some of m A's and n B's. Any such word with i A's and j B's can be extended to a word of $m + 1$ A's and $n + 1$ B's by appending $m + 1 - i$ A's and $n + 1 - j$ B's to it. If our original word begins with an A, we append a block of $m + 1 - i$ A's followed by a block of $n + 1 - j$ B's at the beginning. If the original word begins with a B, we append the block of B's followed by the block of A's at the beginning. The empty word can be extended in two ways: AA...ABB...A or BB...BAA...A. Otherwise, we have a one-to-one correspondence between our original words and words formed from all of $m + 1$ A's and $n + 1$ B's. The reverse correspondence is to take any word of $m + 1$ A's and $n + 1$ B's and delete its first two blocks (i.e., constant subintervals). Since the empty word arises from two extended words, we have

$$\binom{m+n+2}{m+1} - 1$$

of our original words.

As an illustration, let $m = n = 1$:

Original Word	Extended Word
-	AABB or BBAA
A	ABBA
B	BAAB
AB	ABAB

Solution (Partial) by Paul S. Bruckman, Fair Oaks, CA

We let $\underline{n}_r = (n_1, n_2, \ldots, n_r)$ and $\pi(\underline{n}_r) = (n_{i_1}, n_{i_2}, \ldots, n_{i_r})$ denote any permutation of the elements of $\underline{n}_r$. Also, we let

$$S_r(\underline{n}_r) = \sum_{\substack{0 \leqslant i_j \leqslant n_j \\ j=1,2,\ldots,r}} \binom{i_1 + i_2 + \cdots + i_r}{i_1, i_2, \ldots, i_r}, \tag{1}$$

and

$$F_r(\underline{x}_r) = \sum_{\substack{n_j \geqslant 0 \\ j=1,2,\ldots,r}} S_r(\underline{n}_r) x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}, \tag{2}$$

where $\underline{x}_r = (x_1, x_2, \ldots, x_r)$. Then,

$$\begin{aligned} F_r(\underline{x}_r) &= \sum_{\substack{n_j \geqslant 0 \\ j=1,2,\ldots,r}} \sum_{i_j \geqslant 0} \binom{i_1 + i_2 + \cdots + i_r}{i_1, i_2, \ldots, i_r} x_1^{n_1+i_1} x_2^{n_2+i_2} \ldots x_r^{n_r+i_r} \\ &= (1-x_1)^{-1}(1-x_2)^{-1} \ldots (1-x_r)^{-1} \sum_{\substack{i_j \geqslant 0 \\ j=1,2,\ldots,r}} \binom{i_1 + i_2 + \cdots + i_r}{i_1, i_2, \ldots, i_r} x_1^{i_1} x_2^{i_2} \ldots x_r^{i_r} \\ &= (1-x_1)^{-1}(1-x_2)^{-1} \ldots (1-x_r)^{-1} \sum_{n=0}^{\infty} (x_1 + x_2 + \cdots + x_r)^n, \end{aligned}$$

or

$$F_r(\underline{x}_r) = (1 - x_1)^{-1}(1 - x_2)^{-1} \ldots (1 - x_r)^{-1}(1 - x_1 - x_2 - \cdots - x_r)^{-1}. \tag{3}$$

The symmetry inherent in the definition in (1) provides us with the following:

$$S_r(\underline{n}_r) = S_r(\pi(\underline{n}_r)) \text{ for all permutations } \pi. \tag{4}$$

Also, if m is any positive integer less then r, we may set $x_j = 0$ $(m < j \leqslant r)$ in (3) and obtain:

$$S_r(n_1, n_2, \ldots, n_m, \underbrace{0, 0, \ldots, 0}_{r - m}) = S_m(\underline{n}_m). \tag{5}$$

Of course, we may also obtain (5) by setting $n_j = 0$ in (1), $m < j \leqslant r$. Another interesting relationship is obtained by multiplying (3) throughout by the factor $(1 - x_1 - x_2 - \cdots - x_r)$. We then obtain:

$$(1 - x_1)^{-1}(1 - x_2)^{-1} \ldots (1 - x_r)^{-1}$$

$$= (1 - x_1 - x_2 - \cdots - x_r) \sum_{\substack{n_j \geqslant 0 \\ j = 1, 2, \ldots, r}} S_r(\underline{n}_r) x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}.$$

Since

$$(1 - x_1)^{-1}(1 - x_2)^{-1} \ldots (1 - x_r)^{-1} = \sum_{\substack{n_j \geqslant 0 \\ j = 1, 2, \ldots, r}} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r},$$

this yields the recursion:

$$S_r(n_1, n_2, \ldots, n_r) = 1 + S_r(n_1 - 1, n_2, \ldots, n_r) + S_r(n_1, n_2 - 1, \ldots, n_r) + S_r(n_1, n_2, \ldots, n_r - 1). \tag{6}$$

The special cases $r = 1$ and $r = 2$ have already been noted, and are ready consequences of the relations already derived:

$$S_1(n_1) = n_1 + 1; \tag{7}$$

$$S_2(n_1, n_2) = \binom{n_1 + n_2 + 2}{n_1 + 1} - 1. \tag{8}$$

Even for the next case, $r = 3$, however, in spite of the fact that a generating function for the $S_3(n_1, n_2, n_3)$ is known, the general formula is difficult to obtain. By a change of notation, setting $r = 3$ in (6), we obtain:

$$S_3(u, v, w) = 1 + S_3(u - 1, v, w) + S_3(u, v - 1, w) + S_3(u, v, w - 1). \tag{9}$$

The remainder of this manuscript is devoted to the case $r = 3$, and even this is only imperfectly resolved. For brevity in the sequel, the following notation is adopted:

$$U_0 \equiv S_3(u, v, w); \tag{10}$$
$$U_1 \equiv S_3(u - 1, v, w) + S_3(u, v - 1, w) + S_3(u, v, w - 1);$$
$$U_2 \equiv S_3(u, v - 1, w - 1) + S_3(u - 1, v, w - 1) + S_3(u - 1, v - 1, w);$$
$$U_3 \equiv S_3(u - 1, v - 1, w - 1).$$

Thus, a restatement of (9) would be as follows:

$$U_0 = 1 + U_1. \tag{11}$$

Multiplying (2) and (3) throughout by

$$(1 - x)(1 - y)(1 - z)$$

for the case $r = 3$ (by a change of notation), we obtain, on the one hand:

$$(1 - x - y - z)^{-1} = \sum_{u,\, v,\, w \geq 0} \binom{u + v + w}{u,\ v,\ w} x^u y^v z^w.$$

On the other hand, this is equal to

$$(1 - x - y - z + xy + yz + xz - xyz) \sum_{u,\, v,\, w \geq 0} S_3(u,\ v,\ w) x^u y^v z^w.$$

This yields the relation:

$$U_0 - U_1 + U_2 - U_3 = \binom{u + v + w}{u,\ v,\ w}. \tag{12}$$

We use (11) and (12) to derive another interesting recursion involving $S_3(u,\ v,\ w)$:

$$U_0 + U_3 + 1 = \left\{\frac{(u+v+w+1)^2(u+v+w+2) + uvw}{(u+v+1)(v+w+1)(u+w+1)}\right\}\binom{u + v + w}{u,\ v,\ w}. \tag{13}$$

A derivation of (13) follows:

$$\begin{aligned} S_3(u,\ v,\ w) &= \sum_{i=0}^{u} \sum_{j=0}^{v} \sum_{k=0}^{w} \binom{i + j + k}{i,\ j,\ k} = \sum_{i=0}^{u} \sum_{j=0}^{v} \binom{i + j}{i} \sum_{k=0}^{w} \binom{i + j + k}{i + j} \\ &= \sum_{i=0}^{u} \sum_{j=0}^{v} \binom{i + j}{i}\binom{i + j + w + 1}{i + j + 1} \\ &= \sum_{i=0}^{u} \sum_{j=0}^{v} \binom{i + j}{i} \left\{\binom{i + j + w}{i + j} + \binom{i + j + w}{i + j + 1}\right\} \\ &= \sum_{i=0}^{u} \binom{i + w}{w} \sum_{j=0}^{v} \binom{i + w + j}{i + w} + S_3(u,\ v,\ w - 1) \\ &= \sum_{i=0}^{u} \binom{i + w}{w}\binom{i + v + w + 1}{i + w + 1} + S_3(u,\ v,\ w - 1) \\ &= \sum_{i=0}^{u} \binom{i + w}{w}\left\{\binom{i + v + w}{i + w} + \binom{i + v + w}{i + w + 1}\right\} + S_3(u,\ v,\ w - 1) \\ &= \binom{v + w}{v}\sum_{i=0}^{u}\binom{v + w + i}{v + w} + S_3(u,\ v - 1,\ w) \\ &\qquad - S_3(u,\ v - 1,\ w - 1) + S_3(u,\ v,\ w - 1) \\ &= \binom{v + w}{v}\binom{u + v + w + 1}{v + w + 1} + S_3(u,\ v - 1,\ w) \\ &\qquad - S_3(u,\ v - 1,\ w - 1) + S_3(u,\ v,\ w - 1), \end{aligned}$$

or

$$\begin{aligned} &S_3(u,\ v,\ w) + S_3(u,\ v - 1,\ w - 1) \\ &= S_3(u,\ v - 1,\ w) + S_3(u,\ v,\ w - 1) + \frac{(u + v + w + 1)}{(v + w + 1)}\binom{u + v + w}{u,\ v,\ w}. \end{aligned} \tag{14}$$

Interchanging u, v, and w in (14), and adding the resulting relations, we get:

$$3U_0 + U_2 = 2U_1 + \psi\theta, \tag{15}$$

where

$$\psi \equiv (u + v + w + 1)\left\{\frac{1}{u + v + 1} + \frac{1}{v + w + 1} + \frac{1}{u + w + 1}\right\},\quad \theta \equiv \binom{u + v + w}{u,\ v,\ w}.$$

Then, eliminating U_1 and U_2 from (15) by means of (11) and (12) and simplifying the result, we obtain (13).

Applying (8) initially, then (13) recursively with succeeding values of w, we may derive the following formulas:

$$S_3(u,\ v,\ 1) = \frac{(u+1)(v+1)}{(u+v+3)}\binom{u+v+4}{u+2}; \tag{16}$$

$$S_3(u,\ v,\ 2) = \frac{\{(u+1)(v+1)(u+v+5)+4\}(u+v+5)!}{2(u+1)(u+3)(v+1)(v+3)(u+v+3)(u+v+5)u!v!} - 1; \tag{17}$$

$$S_3(u,\ v,\ 3) = \frac{\{(u+2)(v+2)(u+v+5)+12\}(u+v+6)!}{6(u+2)(u+4)(v+2)(v+4)(u+v+3)(u+v+5)u!v!}; \tag{18}$$

$$S_3(u,\ v,\ 4) = \frac{\begin{gathered}(u+1)(u+3)(v+1)(v+3)(u+v+5)(u+v+7)\\ +\ 24(u+1)(v+1)(u+v+7)\ +\ 192\end{gathered}}{24(u+3)(u+5)(v+3)(v+5)(u+v+3)(u+v+5)} \cdot \frac{(u+v+6)!}{(u+1)!(v+1)!}\ 1; \tag{19}$$

$$S_3(u,\ v,\ 5) = \left\{\frac{\begin{gathered}(u+2)(u+4)(v+2)(v+4)(u+v+5)(u+v+7)\\ +\ 40(u+2)(v+2)(u+v+7)\ +\ 960\end{gathered}}{120(u+2)(u+4)(u+6)(v+2)(v+4)(v+6)(u+v+3)(u+v+5)(u+v+7)}\right\} \frac{(u+v+8)!}{u!v!}. \tag{20}$$

From the above formulas, we may infer the following general formula for $S_3(u,\ v,\ w)$:

$$S_3(u,\ v,\ w) = -e_w + \frac{(u+v+w+3)!}{(u+v+3)(u+w+1)(v+w+1)u!v!w!} \cdot \sum_{k=0}^{[\frac{1}{2}w]} \frac{(-1)^k\binom{\frac{1}{2}w}{k}\binom{\frac{1}{2}(w-1)}{k}}{\binom{-\frac{1}{2}(u+v+5)}{k}\binom{\frac{1}{2}(u+w-1)}{k}\binom{\frac{1}{2}(v+w-1)}{k}}, \tag{21}$$

where $e_w = \frac{1}{2}(1+(-1)^w)$.

A proof of (21) was not attempted, although it appears that induction [using (13)] should dispose of it. Unfortunately, the expression in (21) is neither symmetrical in u, v, and w nor in closed form. The more general case, $r \geqslant 3$, appears even more formidable. It seems likely that any fruitful results will require the generating function in (3).

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