A NOTE ON THE CYCLE INDICATOR

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1. INTRODUCTION

Let the cycle indicator

$$C_n(t) \equiv C_n(t_1, \ldots, t_n) = \sum \frac{n!}{k_1! \ldots k_n!}\left(\frac{t_1}{1}\right)^{k_1} \cdots \left(\frac{t_n}{n}\right)^{k_n}, \tag{1}$$

where the summation is over all nonnegative integral values of $k_1, \ldots, k_n$ such that $k_1 + 2k_2 + \cdots + nk_n = n$.

The exponential generating function of $C_n(t)$ is (see [2, Ch. 4]:

$$\exp uC = \sum_{n=0}^{\infty} C_n(t)\frac{u^n}{n} = \exp\left\{\sum_{k=1}^{\infty} \frac{t_k}{k}u^k\right\}, \quad |u| < 1. \tag{2}$$

Applying a Tauberian theorem [1, Th. 5, p. 447] to (2), we will be able to derive a limiting expression of $C_n(t)/n!$, as $n \to \infty$, under certain conditions.

2. A LIMIT THEOREM

Before we state and prove the main theorem, we shall prove the following lemma, which will be useful in the sequel.

Lemma 1

If

$$\frac{1}{n}\sum_{k=1}^{n} t_k \to t, \quad 0 < t < \infty,$$

and the sequence $\{t_n\}$, $n = 1, 2, \ldots$, is monotonic, then the sequence

$$\left\{\frac{C_n(t)}{n!}\right\}, \quad n = 1, 2, \ldots,$$

is monotonic for $n > N$, where N is a fixed number.

Proof: Using the well-known recurrence relation of the cycle indicator, we have:

$$\begin{aligned}\frac{C_{n+1}(t)}{(n+1)!} &= \frac{1}{(n+1)!}\sum_{k=0}^{n} (n)_k\, t_{k+1} C_{n-k}(t)\\ &= \frac{1}{n+1}t_1\frac{C_n(t)}{n!} + \frac{1}{n+1}\left\{t_2\frac{C_{n-1}(t)}{(n-1)!} + \cdots + t_{n+1}\right\}.\end{aligned} \tag{3}$$

Supposing that $\{t_n\}$, $n = 1, 2, \ldots$, is monotonic decreasing, equation (3) is written:

$$\frac{C_{n+1}(t)}{(n+1)!} < \frac{1}{n+1}t_1 \frac{C_n(t)}{n!} + \frac{1}{n+1}\frac{C_n(t)}{n!} = \frac{t_1+1}{n+1}\frac{C_n(t)}{n!}. \tag{4}$$

Since $\{t_n\}$, $n = 1, 2, \ldots$, is bounded, equation (4) is bounded by

$$\frac{C_{n+1}(t)}{(n+1)!} < \left(\frac{N+1}{n+1}\right)\frac{C_n(t)}{n!} \quad \text{or} \quad \frac{C_{n+1}(t)}{(n+1)!} < \frac{C_n(t)}{n!} \quad \text{for all } n > N.$$

Theorem 1

If $\frac{1}{n}\sum_{k=1}^{n} t_k \to t$, as $n \to \infty$, $0 < t < \infty$, then

$$\exp uC \sim \frac{1}{(1-u)^t} L\left(\frac{1}{1-t}\right), \text{ as } u \uparrow 1-, \tag{5}$$

where L is a slowly varying function at infinity.

Furthermore, equation (5) implies that

$$\sum_{k=0}^{n-1} \frac{C_k(t)}{k!} \sim \frac{1}{\Gamma(t+1)} n^t L(n), \text{ as } n \to \infty. \tag{6}$$

If, additionally, $\{t_n\}$, $n = 1, 2, \ldots$, is monotonic, then equation (5) is equivalent to

$$\frac{C_n(t)}{n!} \sim \frac{1}{\Gamma(t)} n^{t-1} L(n), \text{ as } n \to \infty. \tag{7}$$

Proof: Using the relation

$$\sum_{k=1}^{\infty} \frac{u^k}{k} = \log \frac{1}{1-u}, \text{ for } 0 < u < 1,$$

equation (2) is written

$$\exp uC = \frac{1}{(1-u)^t} \exp\left\{\sum_{k=1}^{\infty} \frac{u^k}{k}(t_k - t)\right\}. \tag{8}$$

Letting

$$L\left(\frac{1}{1-u}\right) = \exp\left\{\sum_{k=1}^{\infty} \frac{u^k}{k}(t_k - t)\right\}, \tag{9}$$

and making the substitutions

$$\frac{1}{1-u} = x \quad \text{and} \quad t_k - t = y_k,$$

equation (9) is written

$$L(x) = \exp\left\{\sum_{k=1}^{\infty} \left(1 - \frac{1}{x}\right)^k \frac{y_k}{k}\right\},$$

which is a slowly varying function at infinity, according to [1, Cor. p. 282]. So equation (5) has been proved. Now, applying Theorem 5 of [1, p. 447], we get equation (6).

Using Lemma 1 and the same Theorem we have that (5) is equivalent to (7).

Corollary 1

If

$$\frac{1}{n}\sum_{k=1}^{n} t_k \to t \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^{n} s_k \to s,$$

and the sequences $\{t_n\}$, $\{s_n\}$, $n = 1, 2, \ldots$, are monotonic, then

$$\frac{C_n(t+s)}{n!} \sim \frac{1}{\Gamma(t+s)} n^{t+s-1}L(n), \text{ as } n \to \infty. \tag{10}$$

Proof: Since the $C_n(t)$ is of the binomial type, we have:

$$C_n(t+s) = \sum_{k=0}^{n} \binom{n}{k} C_k(t)C_{n-k}(s). \tag{11}$$

Applying equation (7) to (11), we get

$$\frac{C_n(t+s)}{n!} = \sum_{k=0}^{n} \frac{1}{\Gamma(t)} k^{t-1}L(k)\frac{1}{\Gamma(s)}(n-k)^{s-1}L(n-k) + o(k^{t-1}, (n-k)^{s-1}), \tag{12}$$

where $o(k^{t-1}, (n-k)^{s-1})$ is such that

$$\frac{o(k^{t-1}, (n-k)^{s-1})}{k^{t-1} \cdot (n-k)^{s-1}} \to 0$$

uniformly in k and n as the $\min(k, n-k) \to \infty$.

Equation (12) is equivalent to

$$\frac{C_n(t+s)}{n!} = \frac{n^{t+s-1}}{\Gamma(t)\Gamma(s)} L^2(n) \sum_{\frac{k}{n}=0}^{1} n^{-1}\left(\frac{k}{n}\right)^{t-1}\left(1-\frac{k}{n}\right)^{s-1} \frac{L\left(n\frac{k}{n}\right)}{L(n)} \frac{L\left(n\left(1-\frac{k}{n}\right)\right)}{L(n)}$$
$$+ o\left(\left(\frac{k}{n}\right)^{t-1}, \left(1-\frac{k}{n}\right)^{s-1}\right). \tag{13}$$

By the definition of slowly varying function at infinity, we have that

$$\frac{L\left(n\frac{k}{n}\right)L\left(n\left(1-\frac{k}{n}\right)\right)}{L(n)L(n)} \to 1 \text{ as } n \to \infty.$$

Thus, interpreting the sum in (13) as the approximation to a Riemann integral as $n \to \infty$, we get

$$\frac{C_n(t+s)}{n!} \sim \frac{n^{t+s-1}}{\Gamma(t)\Gamma(s)} L^2(n) \int_0^1 x^{t-1}(1-x)^{s-1}dx$$

or

$$\frac{C_n(t+s)}{n!} \sim \frac{n^{t+s-1}}{\Gamma(t)\Gamma(s)} L(n)B(t, s), \tag{14}$$

where $B(t + s)$ is the Beta function. Since it is well known that

$$B(t,\ s) = \frac{\Gamma(t)\Gamma(s)}{\Gamma(t + s)},$$

equation (14) implies (10).

Corollary 2

If $t_k = t$ for $k = 1, 2, \ldots, 0 < t < 1$, then

$$\sum_{k=0}^{n-1} \frac{C_k(t)}{n!} \sim \frac{1}{\Gamma(t + 1)} n^t, \text{ as } n \to \infty, \tag{15}$$

and

$$\frac{C_n(n)}{n!} \sim \frac{1}{\Gamma(t)} n^{t-1}, \text{ as } n \to \infty. \tag{16}$$

Proof: In this case, the exponential generating function of $C_n(t, \ldots, t)$ is written

$$\exp uC = (1 - u)^{-t} \tag{17}$$

as it is well known [2, p. 70].

Applying Theorem 5 [1, p. 447] to (17) we get (15), and since the sequence

$$\left\{\frac{C_n(t)}{n!}\right\},\ n = 1, 2, \ldots,$$

is monotonic decreasing [2, (11), p. 71], relation (17) is equivalent to (16).

Remark 1: Concerning the same probability problem as that in [2, p. 71], $C_n(t)/n!$ is the generating function of certain probabilities.

Using equation (16), we can easily verify by differentiating that

$$\mu \sim \log(n) + \gamma, \text{ as } n \to \infty,$$

where γ is Euler's constant and

$$\sigma^2 \sim \log n + \gamma + \zeta(2),$$

where $\zeta(2)$ is the Riemann Zeta function which, in this special case, is equal to $\pi^2/6$. Both these results agree with those obtained in [2, p. 72].

REFERENCES

1. W. Feller. *An Introduction to Probability Theory and its Applications.* Vol. II, 2nd ed. New York: Wiley, 1971.
2. J. Riordan. *An Introduction to Combinatorial Analysis.* New York: Wiley, 1958.

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