ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-532 *Proposed by Herta T. Freitag, Roanoke, VA*

Find a, b, and c in terms of n so that

$$a^3(b - c) + b^3(c - a) + c^3(a - b) = 2F_nF_{n+1}F_{n+2}F_{n+3}.$$

B-533 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $g(a, b, c) = a^4(b^2 - c^2) + b^4(c^2 - a^2) + c^4(a^2 - b^2)$. Determine an infinitude of choices for a, b, and c such that $g(a, b, c)$ is the product of five Fibonacci numbers.

B-534 *Proposed by A. B. Patel, V. S. Patel College of Arts & Sciences, Bilimora, India*

One obtains the lengths of the sides of a Pythagorean triangle by letting

$$a = u^2 - v^2,\ b = 2uv,\ c = u^2 + v^2,$$

where u and v are integers with $u > v > 0$. Prove that the area of such a triangle is not a perfect square when $u = F_{n+1}$, $v = F_n$, and $n \geq 2$.

B-535 *Proposed by L. Cseh & I. Merenyi, Cluj, Romania*

Prove that there is no positive integer n for which

$$F_1 + F_2 + F_3 + \cdots + F_{3n} = 16!.$$

<u>B-536</u> *Proposed by L. Kuipers, Sierre, Switzerland*

Find all solutions in integers x and y of

$$x^4 + 2x^3 + 2x^2 + x + 1 = y^2.$$

<u>B-537</u> *Proposed by L. Kuipers, Sierre, Switzerland*

Find all solutions in integers x and y of

$$x^4 + 3x^3 + 3x^2 + x + 1 = y^2.$$

SOLUTIONS

Application of the Bertrand-Chebyshev Theorem

<u>B-508</u> *Proposed by Philip L. Mana, Albuquerque, NM*

Find all n in $\{1, 2, 3, \ldots, 200\}$ such that the sum $n! + (n + 1)!$ of successive factorials is the square of an integer.

I. *Solution by Paul S. Bruckman, Fair Oaks, CA*

Let $\theta_n \equiv n! + (n + 1)! = (n + 2)n!$. We will show that $n = 4$ is the only integer $n \in \{1, 2, 3, \ldots, 200\}$ such that θ_n is square.

<u>Proof</u>: We easily verify that $\theta_1 = 3$, $\theta_2 = 8$, $\theta_3 = 30$, while $\theta_4 = 144 = 12^2$. If $p \leqslant n \leqslant 2p - 3$, where p is any odd prime, then $p|\theta_n$ but $p^2 \nmid \theta_n$; hence, θ_n cannot be a square in this range. Also, if p and q are any two consecutive primes in the sequence of primes, with $5 \leqslant p \leqslant 103$, it is easy to verify that $7 \leqslant q \leqslant 2p - 3 \leqslant 203$. Thus, the range $\{5, 6, 7, \ldots, 200\}$ is spanned by at least one prime p with $p|\theta_n$ but with $p^2 \nmid \theta_n$; this shows that θ_n is not square in this range.

II. *Solution by J. Suck, Essen, Germany*

$n! + (n + !)!$ is a square only for $n = 4$ and a cube only for $n = 2$.

<u>Proof</u>: Bertrand's "postulate" as proved by Chebyshev states that for every integer $k > 3$, there is a prime p satisfying $k < p < 2k - 2$. (See, e.g., Hardy and Wright, *An Introduction to the Theory of Numbers*, 4th ed., p. 373.) Now, let $n = 2m$ or $2m - 1$, $m > 2$. We have a prime p then with $m + 1 < p < 2m$, so that $p|n!$. However, because $2p > 2m + 2 \geqslant n + 2$, p^2 is not a divisor of $n!(n + 2) = n! + (n + 1)!$.

Also solved by L. Cseh, Walther Janous, Edwin M. Klein, L. Kuipers, Imre Merenyi, J. M. Metzger, Bob Prielipp, Neville Robbins, Sahib Singh, M. Wachtel, and the proposer.

Dedekind Function Inequality

<u>B-509</u> *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Let ψ be Dedekind's function given by

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right).$$

For example,

$$\psi(12) = 12(1 + 1/2)(1 + 1/3) = 24.$$

Show that

$$\psi(\psi(\psi(n))) > 2n \text{ for } n = 1, 2, 3, \ldots .$$

Solution by J. M. Metzger, University of N. Dakota, Grand Forks, ND

The statement is false for $n = 1$.

Since $\psi(\psi(\psi(2))) = 6$, the inequality is correct for $n = 2$.

Now assume $n \geqslant 3$. For such n, $\psi(n)$ is clearly even. Note that for all $n \geqslant 2$, $\psi(n) \geqslant n + 1$ because $\psi(n)$ is an integer greater than n. Moreover, if k is even, then

$$\psi(k) = k \prod_{p|k} \left(1 + \frac{1}{p}\right) \geqslant k \cdot \left(1 + \frac{1}{2}\right) = \frac{3k}{2}.$$

It follows that

$$\psi(\psi(\psi(n))) \geqslant \frac{3}{2}\,\psi(\psi(n)) \geqslant \frac{3}{2}\cdot\frac{3}{2}\,\psi(n) \geqslant \frac{9}{4}(n + 1) > 2n.$$

Also solved by Paul S. Bruckman, L. Cseh, Alberto Facchini, C. Georghiou, Walther Janous, L. Kuipers, I. Merenyi, Lawrence Somer, J. Suck, and the proposer.

Inequality on Euler and Dedekind Functions

B-510 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Euler's ϕ function and its companion, Dedekind's ψ function, are defined by

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \quad \text{and} \quad \psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right).$$

(a) Show that $\phi(n) + \psi(n) \geqslant 2n$ for $n > 1$.
(b) When is the inequality strict?

Solution by Alberto Facchini, University of Udine, Italy

Let $p_1, \ldots, p_t$ be the prime factors of n. Then,

$$\begin{aligned}\prod_{p|n}\left(1 \pm \frac{1}{p}\right) = 1 &\pm \left(\frac{1}{p_1} + \cdots + \frac{1}{p_t}\right)\\ &+ \left(\frac{1}{p_1p_2} + \frac{1}{p_1p_3} + \cdots + \frac{1}{p_1p_t} + \frac{1}{p_2p_3} + \cdots + \frac{1}{p_{t-1}p_t}\right)\\ &\pm \left(\frac{1}{p_1p_2p_3} + \cdots\right) + \cdots + (\pm 1)^t \frac{1}{p_1p_2 \cdots p_t}.\end{aligned}$$

Therefore,

$$\phi(n) + \psi(n) = 2n\left[1 + \left(\frac{1}{p_1p_2} + \frac{1}{p_1p_3} + \cdots + \frac{1}{p_{t-1}p_t}\right) + \cdots\right] \geqslant 2n$$

and the inequality is strict if and only if n has at least two distinct prime factors.

Also solved by Paul S. Bruckman, L. Cseh, C. Georghiou, Walther Janous, L. Kuipers, Vania D. Mascioni, I. Merenyi, J. M. Metzger, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Telescoping Fibonacci Products

B-511 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers with j even. Prove that

$$F_j(F_n + F_{n+2j} + F_{n+4j} + \cdots + F_{n+2jk}) = (L_{n+2jk+j} - L_{n-j})/5.$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We shall show that

$$5F_nF_j + 5F_{n+2j}F_j + 5F_{n+4j}F_j + 5F_{n+6j}F_j + \cdots + 5F_{n+(2k-2)j}F_j + 5F_{n+2jk}F_j$$
$$= L_{n+(2k+1)j} - L_{n-j},$$

which is clearly equivalent to the desired result. From (12) on p. 115 of the April 1975 issue of this journal,

$$5F_sF_t = L_{s+t} - L_{s-t},\ t \text{ even}.$$

Thus, since j is even,

$$5F_nF_j + 5F_{n+2j}F_j + 5F_{n+4j}F_j + 5F_{n+6j}F_j + \cdots + 5F_{n+(2k-2)j}F_j + 5F_{n+2jk}F_j$$
$$= (L_{n+j} - L_{n-j}) + (L_{n+3j} - L_{n+j}) + (L_{n+5j} - L_{n+3j}) + (L_{n+7j} - L_{n+5j})$$
$$+ \cdots + (L_{n+(2k-1)j} - L_{n+(2k-3)j}) + (L_{n+(2k+1)j} - L_{n+(2k-1)j})$$
$$= L_{n+(2k+1)j} - L_{n-j}$$

because telescoping occurs.

Also solved by Paul S. Bruckman, L. Cseh, Herta T. Freitag, C. Georghiou, Walther Janous, L. Kuipers, I. Merenyi, H.-J. Seiffert, A. G. Shannon, J. Suck, Sahib Singh, and the proposer.

Telescoping Fibonacci-Lucas Products

B-512 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers with j odd. Prove that

$$L_j(F_n + F_{n+2j} + F_{n+4j} + \cdots + F_{n+2kj}) = F_{n+2kj+j} - F_{n-j}.$$

Solution by J. Suck, Essen, Germany

Do not use induction. Just telescope the left-hand side by Hoggatt's I_{23}:

$$L_jF_m = F_{m+j} - F_{m-j},\ j \text{ odd}.$$

Also solved by Paul S. Bruckman, L. Cseh, Herta T. Freitag, C. Georghiou, Walther Janous, L. Kuipers, I. Merenyi, Bob Prielipp, H.-J. Seiffert, A.G. Shannon, Sahib Singh, and the proposer.

Fibonacci Convolution and Rising Pascal Diagonals

B-513 *Proposed by Andreas N. Philippou, University of Patras, Greece*

Show that

$$\sum_{k=0}^{n} F_{k+1}F_{n+1-k} = \sum_{k=0}^{[n/2]} (n + 1 - k)\binom{n-k}{k} \text{ for } n = 0, 1, \ldots,$$

where $[x]$ denotes the greatest integer in x.

Solution by C. Georghiou, University of Patras, Greece

Since the generating function of the sequence $\{F_{n+1}\}_{n=0}^{\infty}$ is

$$f(x) = (1 - x - x^2)^{-1},$$

it follows that

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{n} F_{k+1}F_{n+1-k}x^n &= (1 - x - x^2)^{-2} \\
&= \sum_{n=0}^{\infty} (-1)^n\binom{-2}{n}(x + x^2)^n, \quad |x| < \frac{1}{\alpha} \\
&= \sum_{n=0}^{\infty} (n + 1)(x + x^2)^n \\
&= \sum_{n=0}^{\infty} \sum_{j+k=n} (j + k + 1)\binom{j+k}{k}x^{j+2k} \\
&= \sum_{n=0}^{\infty} \sum_{j+2k=n} (j + k + 1)\binom{j+k}{k}x^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} (n + 1 - k)\binom{n-k}{k}x^n
\end{aligned}$$

from which the assertion is established.

Also solved by Paul S. Bruckman, L. Cseh, Walther Janous, L. Kuipers, H.-J. Seiffert, A. G. Shannon, J. Suck, and the proposer.

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