AN ORDER-THEORETIC REPRESENTATION OF THE POLYGONAL NUMBERS

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1. INTRODUCTION

Polygonal numbers received their name from their standard geometric realization. In this geometric realization one considers sequences of regular polygons that share a common angle and have points at equal distances along each side. The total number of points on a sequence of the regular polygons is a sequence of polygonal numbers. For example (see Fig. 1), if the polygon is a triangle, we get the triangular numbers 1, 3, 6, 10, 15, ..., and if the polygon is a pentagon, we get the pentagonal numbers 1, 5, 12, 22, 35, ... More information on the polygonal numbers may be found in L. E. Dickson's *History of Number Theory* [4, Vol. II, Ch. 1]. We also recommend the discussion of "figurirte oder vieleckigte Zahlen" by L. Euler [5, p. 159].

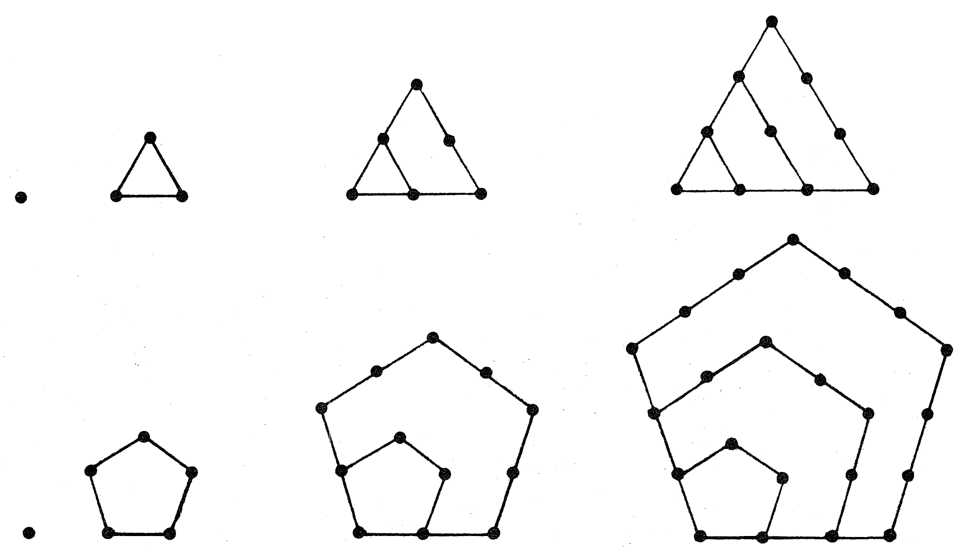


FIGURE 1

In this paper we describe an order-theoretic realization of the polygonal numbers. We represent the polygonal numbers as the cardinalities of sequences of modular lattices that can be glued together from simple building blocks. The construction of these lattices is described in the first part of §3, the main result is formulated in Theorem 3.3. It is interesting to note that, in the case of the triangular numbers and of the square numbers, the diagrams in our lattice-theoretic representation (Figure 3 and Figure 4) look just like the usual illustrations in the standard geometric realization. For all other polygonal numbers, however, the diagrams are very different.

In §2 we introduce some essential terminology and necessary facts about function lattices. For a more complete treatment of these topics, we refer the reader to the standard textbooks [1], [2], [6], and to [3].

2. FUNCTION LATTICES

Let P and Q be partially ordered sets. A mapping $f:P \to Q$ is order-preserving if $x \leqslant y$ in P implies $f(x) \leqslant f(y)$ in Q for all x, $y \in P$. An order-isomorphism is a mapping f that is one-to-one, onto, and has the property that $x \leqslant y$ in P if and only if $f(x) \leqslant f(y)$ in Q, for all x, $y \in P$. The set Q^P of all the order-preserving mappings from P to Q can be partially ordered by $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all $x \in P$. If f, $g \in Q^P$, then the supremum of f and g, $f \vee g$, exists in Q^P if and only if the supremum of $f(x)$ and $g(x)$ exists in Q for all $x \in P$, and $(f \vee g)(x) = f(x) \vee g(x)$. Since the same is true for the infimum of f and g, it follows that Q^P is a lattice whenever Q is a lattice, P may be an arbitrary partially ordered set. It can be shown that the function lattice Q^P is a distributive or modular lattice provided that Q is a distributive or modular lattice, respectively.

For integers $n \geqslant 0$, $\underline{n} = \{1, 2, \ldots, n\}$ denotes the totally ordered chain of n elements ordered in their natural order, $\underline{0}$ the empty chain, and $\underline{m}^{\underline{n}}$ the distributive function lattice of order-preserving mappings from the n-element chain $\underline{n}$ into the m-element chain $\underline{m}$. $M(n)$ is the modular lattice of length 2 with n atoms, $M(0) = \underline{2}$, $M(1) = \underline{3}$.

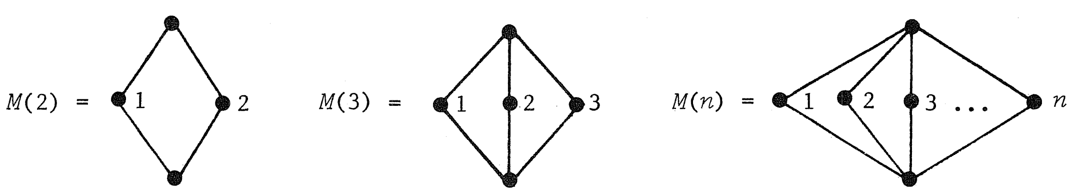


FIGURE 2

An element a in a lattice is join-irreducible if $a = b \vee c$ implies $a = b$ or $a = c$; it is meet-irreducible if $a = b \wedge c$ implies $a = b$ or $a = c$. A doubly irreducible element is an element which is both join- and meet-irreducible. Chains of doubly irreducible elements will play an important role in the construction in 3. As examples we shall now determine the sequences of function lattices $\underline{3}^{\underline{n}} = M(1)^{\underline{n}}$ (Fig. 3) and $M(2)^{\underline{n}}$ (Fig. 4) for $n \geqslant 0$. In $\underline{3}^{\underline{n}}$, the doubly irreducible elements are circled where the function $f:\underline{n} \to \underline{3}$ is represented by its image vector, i.e., 1223 stands for the function $f:\underline{4} \to \underline{3}$ given by $f(1) = 1$, $f(2) = f(3) = 2$, and $f(4) = 3$.

Obviously, the cardinalities of the lattices in Figure 3 are the triangular numbers, the cardinalities of the lattices in Figure 4 are the square numbers. This, of course, raises the question: Is it possible to represent all polygonal numbers as function lattices?

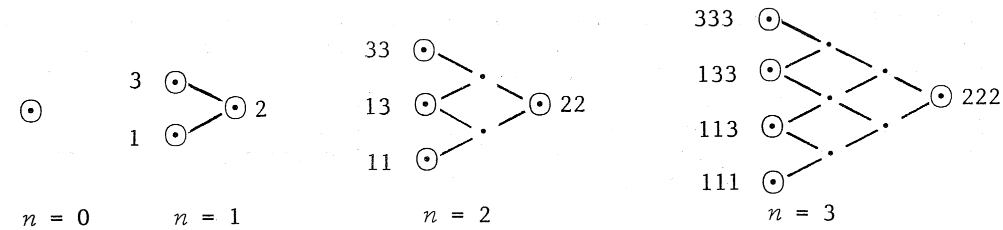


FIGURE 3

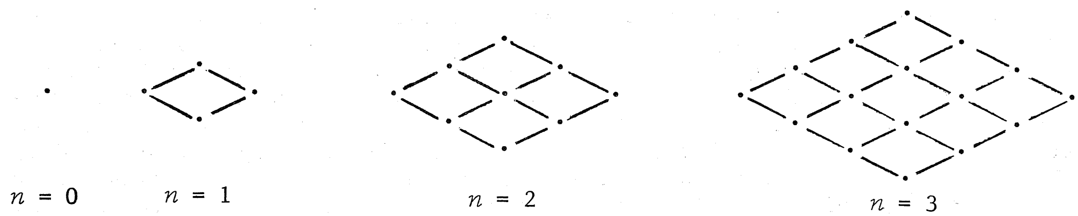


FIGURE 4

3. MODULAR PADDLEWHEELS AND POLYGONAL NUMBERS

Let $C = \{c_0 < c_1 < \cdots < c_n\}$ be a chain and let L_i, $1 \leqslant i \leqslant k$, be partially ordered sets with least and largest elements, z_i and e_i, which admit order-isomorphisms $\phi : C \to L_i$ into L_i so that $\phi_i(c_0) = z_i$ and $\phi_i(c_n) = e_i$ for each i. On the disjoint union of the L_i, $1 \leqslant i \leqslant k$, we define a relation R by $(x, y) \in R$ if and only if

$$\phi_i^{-1}(x) = \phi_j^{-1}(y) \text{ for some } i \text{ and } j, \text{ or } x = y.$$

R is an equivalence relation and the factorization of $\cup\{L_i \mid 1 \leqslant i \leqslant k\}$ with respect to this equivalence relation, denoted by $M = M(L_1, \ldots, L_k; C)$, is a partially ordered set where the order of each piece L_i is the given order, and if $x \in L_i$ and $y \in L_j$, $i \neq j$, then $x \leqslant_M y$ if and only if there is $0 \leqslant s \leqslant n$ so that $x \leqslant \phi_i(c_s)$ and $\phi_j(c_s) \leqslant y$. Moreover, if we let

$$m = \min\{t \mid x \leqslant \phi_i(c_t) \text{ and } y \leqslant \phi_j(c_t)\},$$

then either $\quad x \nleqslant \phi_i(c_{m-1}) \quad$ and $\quad y \nleqslant \phi_j(c_{m-1})$

or $\quad x \leqslant \phi_i(c_{m-1}) \quad$ and $\quad y \nleqslant \phi_j(c_{m-1})$

or $\quad x \nleqslant \phi_i(c_{m-1}) \quad$ and $\quad y \leqslant \phi_j(c_{m-1})$.

In the first case, $x \vee_M y = \phi_i(c_m)$ holds. In the second case, any common upper bound $z \in M$ of x and y such that $z \ngeqslant \phi_i(c_m)$ is in the piece L_j; hence, $x \vee_M y$ exists in the piece L_j if L_j has suprema. In the third case, $x \vee_M y$ exists in L_i if suprema exist in L_i. Of course $x \wedge_M y$ behaves in a similar way. Therefore, $M = M(L_1, \ldots, L_k; C)$ is a lattice whenever each L_i is a lattice.

We will use this construction only in the case where $L_i = L_j$ and $\phi_i = \phi_j$, for all i and j, and we indicate that we have k copies of the same lattice L in the abbreviated notation $M = M(k(L); C)$.

If all $L_i = L$, a three-dimensional illustration of $M(k(L); C)$ looks like a paddlewheel with k paddles, with the chain C as the vertical axis, and the k copies of the lattice L as the paddles, equally spaced around a circle and glued to the chain C by the mappings $\phi = \phi_i$, for all $1 \leqslant i \leqslant k$.

As an example, let

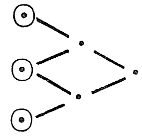
$L = \underline{3}^{\underline{2}} =$ 

FIGURE 5

We want to construct $M(4(L); 3)$. $\underline{3}^{\underline{2}}$ contains an order-isomorphic copy of $\underline{3}$, namely, a three-element chain of doubly irreducible elements, circled in the diagram above. Four copies of L are glued to this chain and we get

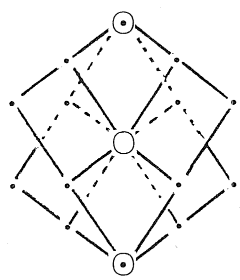


FIGURE 6

The following theorem will show that this lattice is $M(4)^{\underline{2}}$. The proof of the theorem requires some knowledge of the irreducible elements in $\underline{3}^{\underline{n}}$.

Every function $f:\underline{n} \to \underline{m}$ is piecewise constant and may be written as an increasing tuple of m values. A convenient notation is

$$1^{k_1} 2^{k_2} \ldots m^{k_m}$$

with $k_i \geqslant 0$, $1 \leqslant i \leqslant m$, and $k_1 + \cdots + k_m = n$, where the exponents k_i count the number of occurrences of the value i for the function $f(x) = i$ if and only if

$$k_1 + \cdots + k_{i-1} < x \leqslant k_1 + \cdots + k_i.$$

Now, there are two types of doubly irreducible elements in $\underline{m}^{\underline{n}}$, the constant mappings where $k_i = n$ for exactly one i, and $k_j = 0$ for all $j \neq i$, and those whose only values are the extremal elements of $\underline{m}$. The latter are of the form

$$1^{k_1} m^{k_m}, \text{ where } k_i = 0 \text{ for all } 1 < i < m.$$

The constant mappings obviously form a chain of m elements in $\underline{m}^{\underline{n}}$. For the second type of doubly irreducible elements, we have $k_1 + k_m = n$; hence, the possibilites $k_m = 0, 1, \ldots, n$, and therefore these doubly irreducible elements form a chain of $n + 1$ elements in $\underline{m}^{\underline{n}}$. This is the chain that we want to use for our paddlewheel construction. So in Theorem 3.1, $\underline{n} \oplus \underline{1}$ may be interpreted as the chain of these doubly irreducible elements in $\underline{3}^{\underline{n}}$, with $\phi:\underline{n} \oplus \underline{1} \to \underline{3}^{\underline{n}}$ the identity mapping.

Theorem 3.1

$M = M(k(\underline{3}^{\underline{n}});\ \underline{n} \oplus \underline{1})$ is the modular lattice $M(k)^{\underline{n}}$, for $k \geq 1$, $n \geq 0$.

Proof: An element in $\underline{3}^{\underline{n}}$ may be represented as $z^\alpha u^\beta e^\gamma$, where $\alpha + \beta + \gamma = n$, $0 \leq \alpha, \beta, \gamma \leq n$, and where $z < u < e$ is the chain $\underline{3}$. Similarly, we represent elements in M as $(z^\alpha u^\beta e^\gamma)_i$ for $1 \leq i \leq k$, where the index i indicates that the element is in the i^{th} of the k copies of $\underline{3}^{\underline{n}}$. Elements in $M(k)^{\underline{n}}$ are of the form $z^p j^r e^t$, where $p + r + t = n$, $0 \leq p, r, t \leq n$, and j is the j^{th} of the k atoms of $M(k)$.

We now define a mapping $\psi : M \to M(k)^{\underline{n}}$ by

$$\psi((z^\alpha u^\beta e^\gamma)_i) = z^\alpha i^\beta e^\gamma.$$

Should $(z^\alpha u^\beta e^\gamma)_i$ be in the chain $\underline{n} \oplus \underline{1}$ of M, i.e., i is not uniquely determined, then it is doubly irreducible with $\beta = 0$ and its image under ψ is then of the form $z^\alpha e^\gamma$ with $\alpha + \gamma = n$; in other words, it is independent of i. ψ is thus well defined, and it is rather straightforward to show that ψ is an order-isomorphism. ■

Note that for $k = 1$ we have

$$M(1(\underline{3}^{\underline{n}});\ \underline{n} \oplus \underline{1}) \simeq \underline{3}^{\underline{n}} \text{ (see Fig. 3)},$$

and for $k = 2$ we have

$$M(2(\underline{3}^{\underline{n}});\ \underline{n} \oplus \underline{1}) \simeq M(2)^{\underline{n}}.$$

In the latter case, the two copies of $\underline{3}^{\underline{n}}$ are glued together so that we get a planar diagram symmetric on its vertical axis (see Fig. 4). This representation theorem provides a procedure to calculate $|M(k)^{\underline{n}}|$, the number of elements of $M(k)$, from the number of elements in $\underline{3}^{\underline{n}}$. But

$$|\underline{3}^{\underline{n}}| = \binom{n+2}{n},$$

which can be easily verified by induction on n.

Theorem 3.2

$$|M(k)^{\underline{n}}| = \left(k \cdot \frac{n}{2} + 1\right) \cdot (n + 1) \text{ for all } n,\ k > 0.$$

Proof: For $k = 0$, $|M(0)^{\underline{n}}| = |\underline{2}^{\underline{n}}| = n + 1$. In all other cases, it follows from the representation in Theorem 3.1,

$$M(k)^{\underline{n}} = M(k(\underline{3}^{\underline{n}});\ \underline{n} \oplus \underline{1}),$$

that $|M(k)^{\underline{n}}| = k \cdot |\underline{3}^{\underline{n}}| - (k - 1) \cdot (n + 1)$. Since

$$|3^{\underline{n}}| = \binom{n+2}{n},$$

we get

$$|M(k)^{\underline{n}}| = k \cdot \binom{n+2}{n} - (k - 1) \cdot (n + 1) = \left(k \cdot \frac{n}{2} + 1\right) \cdot (n + 1). \ ■$$

It is now easy to see that the numbers $P_{n,k} = |M(k)^{\underline{n}}|$ also satisfy the recursion formula

$$\begin{aligned} P_{n,k} &= P_{n,k-1} + P_{n-1,1} && \text{for } n,\ k > 0 \\ P_{n,0} &= n + 1 && \text{for } n \geq 0 \\ P_{0,k} &= 1 && \text{for } k \geq 0. \end{aligned}$$

However, this recursion defines the polygonal numbers [4, Vol. II, Ch. 1]. So we find that the modular lattices $M(k)^{\underline{n}}$ are order-theoretic realizations of the polygonal numbers.

Theorem 3.3

The cardinalities of the sequence of modular lattices $M(k)^{\underline{n}}$ for increasing $n \geqslant 0$ and for $k \geqslant 0$ are the polygonal numbers.

To illustrate the connection between $|M(k)^{\underline{n}}|$ and polygonal numbers, we list them in the following table for $n, k \leqslant 5$. For example, the horizontal line with entry $k = 3$ contains, from left to right, the numbers

$$1 = |M(3)^{\underline{0}}|,\ 5 = |M(3)^{\underline{1}}|,\ 12 = |M(3)^{\underline{2}}|, \text{ etc.}$$

These are the pentagonal numbers, listed in [7] as sequence number 1562.

$k \backslash n$	0	1	2	3	4	5	Name	Sloan Number
0	1	2	3	4	5	6	natural numbers	# 173
1	1	3	6	10	15	21	triangular numbers	# 1002
2	1	4	9	16	25	36	squares	# 1350
3	1	5	12	22	35	51	pentagonal numbers	# 1562
4	1	6	15	28	45	66	hexagonal numbers	# 1705
5	1	7	18	34	55	81	heptagonal numbers	# 1826

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