PELL NUMBERS AND COAXAL CIRCLES

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1. INTRODUCTION

The purpose of this note is to generalize the results in [2] and to apply them to the particular case of Pell numbers. An acquaintance with [2] is desirable.

Define the generalized sequence $\{W_n\}$ by

$$W_n = pW_{n-1} - qW_{n-2},\ W_0 = r,\ W_1 = r + s \tag{1.1}$$

for all integral n, where p, q, r, and s are arbitrary, but will generally be thought of as integers.

Then, from [1], *mutatis mutandis*,

$$W_n = \frac{(r + s - r\beta)\alpha^n - \{(r + s) - r\alpha\}\beta^n}{\Delta}, \tag{1.2}$$

where α and β are the roots of $x^2 - px + q = 0$, so that $\alpha + \beta = p$, $\alpha\beta = q$, and $\alpha - \beta = \Delta = \sqrt{p^2 - 4q}$.

The generalized sequence $\{H_n\}$ in [2] occurs when

$$p = 1,\ q = -1,\ \Delta = \sqrt{5},\ r = 2b,\ \text{and } s = a - b,$$

with the special cases of the Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ arising when $a = 1$, $b = 0$ (i.e., $r = 0$, $s = 1$) and $a = 0$, $b = 1$ (i.e., $r = 2$, $s = -1$), respectively.

Our particular concern in this note is with the case $p = 2$, $q = -1$, where $\alpha = 1 + \sqrt{2}$ (>0), $\beta = 1 - \sqrt{2}$ (<0), i.e., $\Delta = 2\sqrt{2}$.

Writing W_n' for W_n when $p = 2$, $q = -1$, we have from (1.2) that

$$W_n' = sP_n + \frac{r}{2}Q_n, \tag{1.3}$$

where

$$P_n = (\alpha^n - \beta^n)/2\sqrt{2} \tag{1.4}$$

and

$$Q_n = \alpha^n + \beta^n \tag{1.5}$$

are the n^{th} Pell and the n^{th} "Pell-Lucas" numbers, respectively, occurring in (1.1), (1.2), and (1.3) when $r = 0$, $s = 1$ (for P_n) and $r = 2$, $s = 0$ (for Q_n).

From (1.4) and (1.5), we have

$$2\sqrt{2}P_n < Q_n \quad \text{when } n \text{ is even,} \tag{1.6}$$

and

$$2\sqrt{2}P_n > Q_n \quad \text{when } n \text{ is odd.} \tag{1.7}$$

2. COAXAL CIRCLES FOR $\{W_n\}$

Consider the point $(x, 0)$ in the Euclidean plane with

$$x = [(r + s - r\beta)\alpha^{2n} + (-(r + s) + r\alpha)\cos(n - 1)\pi]/\Delta\alpha^n \tag{2.1}$$

The circle CW_n having

$$\text{center} \quad \bar{x}(W_n) = \frac{(r + s - r\beta)}{\Delta}\alpha^n, \quad \bar{y}(W_n) = 0, \tag{2.2}$$

and

$$\text{radius} \quad r(W_n) = \left|\frac{-(r + s) + r\alpha)}{\Delta\alpha^n}\right| \tag{2.3}$$

has the equation

$$\left(x - \frac{(r + s - r\beta)}{\Delta}\alpha^n\right)^2 + y^2 = \left(\frac{-(r + s) + r\alpha}{\Delta\alpha^n}\right)^2, \tag{2.4}$$

so that

$$\bar{x}(W_n)/\bar{x}(W_{n-1}) = \alpha \tag{2.5}$$

and

$$r(W_n)/r(W_{n-1}) = \frac{1}{\alpha}. \tag{2.6}$$

The points of intersection of CW_n and the x-axis are given by

$$\begin{aligned} x(W_n) &= \frac{(r + s - r\beta)\alpha^n}{\Delta} \pm \frac{(-(r + s) + r\alpha)}{\Delta\alpha^n} \\ &= \left\{(r + s)\left\{\alpha^n \mp \frac{\beta^n}{q^n}\right\} - rq\left\{\alpha^{n-1} \mp \frac{\beta^{n-1}}{q^n}\right\}\right\}\Big/\Delta. \end{aligned} \tag{2.7}$$

Highest points on CW_n lie on the upper branch of the rectangular hyperbola

$$xy = (r + s - r\beta)\,|(r + s - r\alpha)|/\Delta^2.$$

3. COAXAL CIRCLES FOR $\{P_n\}$ AND $\{Q_n\}$

Proceeding now to the Pell numbers P_n (1.4) and Pell-Lucas numbers Q_n (1.5) we can tabulate results corresponding to the more general results (2.1)-(2.8) as follows.

Eq.	$\{P_n\}$	$\{Q_n\}$
(3.1)	$\begin{cases} x = \{\alpha^{2n} - \cos(n-1)\pi\}/2\sqrt{2}\alpha^n \\ y = 0 \end{cases}$	$\begin{cases} x = \{\alpha^{2n} + \cos(n-1)\pi\}/\alpha^n \\ y = 0 \end{cases}$
(3.2)	$\bar{x}(P_n) = \alpha^n/2\sqrt{2},\ \bar{y}(P_n) = 0$	$\bar{x}(Q_n) = \alpha^n,\ \bar{y}(Q_n) = 0$
(3.3)	$r(P_n) = 1/2\sqrt{2}\alpha^n$	$r(Q_n) = 1/\alpha^n$
(3.4)	$CP_n\colon \left\{x - \dfrac{\alpha^n}{2\sqrt{2}}\right\}^2 + y^2 = \dfrac{1}{8\alpha^{2n}}$	$CQ_n\colon (x - \alpha^n)^2 + y^2 = \dfrac{1}{\alpha^{2n}}$
(3.5)	$\bar{x}(P_n)/\bar{x}(P_{n-1}) = \alpha$	$\bar{x}(Q_n)/\bar{x}(Q_{n-1}) = \alpha$
(3.6)	$r(P_n)/r(P_{n-1}) = \dfrac{1}{\alpha}$	$r(Q_n)/r(Q_{n-1}) = \dfrac{1}{\alpha}$
(3.7)	$x(P_n) = P_n,\ \dfrac{Q_n}{2\sqrt{2}}$	$x(Q_n) = Q_n,\ 2\sqrt{2}P_n$
(3.8)	$xy = \dfrac{1}{8}$	$xy = 1$

Remarks about the circle-generation of Pell and Pell-Lucas numbers, similar to those made about results (3.7) in the tabulation in [2], may now be made about results (3.7) in the preceding table.

It is worth noting that the same locus $xy = 1$ in (3.8) arises from both the Lucas numbers L_n [2] and the Pell-Lucas numbers Q_n, although the two sequences of points on the hyperbola are different.

There do not appear to be any really interesting geometrical relations among the circles associated with F_n, L_n, P_n, and Q_n.

In passing, we note that in (3.7) we use

$$P_n + P_{n-1} = \frac{1}{2}Q_n$$

$$Q_n + Q_{n-1} = 4Q_n$$

both of which may be easily derived from (1.4) and (1.5).

REFERENCES

1. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* 3, no. 3 (1965):161-76.
2. A. F. Horadam. "Coaxal Circles Associated with Recurrence-Generated Sequences." *The Fibonacci Quarterly* 22, no. 3 (1984):270-72, 278.

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