SOME PREDICTABLE PIERCE EXPANSIONS

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I. INTRODUCTION

In 1929, T. A. Pierce discussed an algorithm for expanding real numbers $x \in (0, 1)$ in the form

$$x = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} \cdots, \tag{1}$$

where the a_i form a strictly increasing sequence of positive integers.

He showed that these expansions (which we call *Pierce expansions*) are essentially unique. The Pierce expansion for x terminates if and only if x is rational. See [3] and [5] for details.

In this note, we give formulas for the a_i in the case where

$$x = \frac{c - \sqrt{c^2 - 4}}{2}$$

and $c \geqslant 3$ is an integer. For these numbers, Pierce expansions provide extremely rapidly converging series.

II. FINDING REAL ROOTS OF POLYNOMIALS

To save space, we sill sometimes write equation (1) in the form

$$x = \{a_1, a_2, a_3, \ldots\},$$

where the braces denote a Pierce expansion.

Let

$$p_1(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

be a polynomial with integer coefficients and a single real zero α in the interval (0, 1). We want to find the first term in the Pierce expansion of α. From equation (1) it is easy to see that $a_1 = \lfloor 1/\alpha \rfloor$. Consider the polynomial $q_1(x) = x^n p_1(1/x)$; this is a polynomial with integer coefficients that has $1/\alpha$ as a zero. Through a simple binary search procedure, it is easy to find d_1 such that

$$\text{sign}(q_1(d_1)) = \text{sign}(q_1(d_1 + 1));$$

this shows that $d_1 = \lfloor 1/\alpha \rfloor$ and so we can take $\alpha_1 = d_1$.

Now consider the polynomial

$$p_2(x) = a_1^n p_1\left(\frac{1 - x}{a_1}\right).$$

This again is a polynomial with integer coefficients. It is easily verified that if β is a zero of $p_2(x)$, then

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1} \beta$$

so

$$\beta = \frac{1}{a_2} - \frac{1}{a_2 a_3} + \cdots .$$

By repeating this procedure on the polynomial $p_2(x)$, we generate the coefficient α_2 in the Pierce expansion of α, and by continuing in the same fashion, we can generate as many terms of the Pierce expansion for α as desired:

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots .$$

Now let us specify our polynomial to be

$$p\ (x) = x^2 - cx + 1,$$

where $c \geqslant 3$ is an integer. Let α be the smaller positive zero, so

$$\alpha = \frac{c - \sqrt{c^2 - 4}}{2}. \tag{2}$$

Now $q_1(x) = x^2 p_1(1/x) = x^2 - cx + 1$. We find $q_1(c - 1) = 2 - c$, which is negative, and $q_1(c) = 1$, which is positive. Hence, we see that $a_1 = c - 1$.

Now

$$p_2(x) = (c - 1)^2 p_1\left(\frac{1 - x}{c - 1}\right);$$

hence,

$$p_2(x) = x^2 + (c^2 - c - 2)x + 2 - c.$$

We find

$$q_2(x) = x^2 p_2(1/x) = (2 - c)x^2 + (c^2 - c - 2)x + 1.$$

Now $q_2(c + 1) = 1$, which is positive; but $q_2(c + 2) = 5 - c^2$, which is negative. Hence, we see that $a_2 = c + 1$.

Now

$$p_3(x) = x^2 p_2\left(\frac{1 - x}{c + 1}\right),$$

so we see

$$p_3(x) = x^2 - (c^3 - 3c)x + 1.$$

So far we have been following the algorithm. But now we notice that $p_3(x)$ is essentially just $p_1(x)$ with $c^3 - 3c$ playing the role of c. We have found

$$\alpha = \frac{1}{c - 1} - \frac{1}{(c - 1)(c + 1)} + \frac{1}{(c - 1)(c + 1)}\gamma,$$

where γ is the root of $x^2 - (c^3 - 3c)x + 1 = 0$. By continuing this process, we get:

<u>Theorem</u>

Let α be as in equation (2). Then,

$$\alpha = \{c_0 - 1,\ c_0 + 1,\ c_1 - 1,\ c_1 + 1,\ c_2 - 1,\ c_2 + 1,\ \ldots\},$$

where $c_0 = c$, $c_{k+1} = c_k^3 - 3c_k$.

For example, let $c = 3$. Then we find

$$\frac{3 - \sqrt{5}}{2} = \{2, 4, 17, 19, 5777, 5779, \ldots\}.$$

Another example: let $c = 6$. Then, after some manipulation, we find

$$\sqrt{2} - 1 = \{2, 5, 7, 197, 199, 7761797, 7761799, \ldots\}.$$

Ironically, both Pierce [3] and Salzer [4] gave the first four terms of this expansion, but apparently neither detected the general pattern!

III. THE COEFFICIENTS c_k

The recurrence $c_{k+1} = c_k^3 - 3c_k$ is an interesting one which has been previously studied ([1], [2]). Some brief comments are in order.

If we let α and β be the roots of the quadratic

$$x^2 - cx + 1 = 0,$$

with $\alpha < \beta$, and define

$$V(n) = \alpha^n + \beta^n; \; U(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

then it is easy to show by induction that

$$V(n) = cV(n - 1) - V(n - 2); \; U(n) = cU(n - 1) - U(n - 2),$$

where

$$V(0) = 2, \; V(1) = c; \; U(0) = 0, \; U(1) = 1.$$

We can also show that $V(3k) = V(k)^3 - 3V(k)$; hence, by induction, $c_k = V(3^k)$. This gives the following closed form for the c_k:

$$c_k = \left(\frac{c + \sqrt{c^2 - 4}}{2}\right)^{3^k} + \left(\frac{c - \sqrt{c^2 - 4}}{2}\right)^{3^k}.$$

Similarly, it can be shown by induction that

$$\frac{U(3^k - 1)}{U(3^k)} = \{c_0 - 1, c_0 + 1, c_1 - 1, c_1 + 1, \ldots, c_{k-1} - 1, c_{k-1} + 1\}. \quad (3)$$

Here is a sketch of the induction step. Assuming (3) holds, we find

$$\{c_0 - 1, c_0 + 1, c_1 - 1, c_1 + 1, \ldots, c_k - 1, c_k + 1\}$$

$$= \frac{U(3^k - 1)}{U(3^k)} + \frac{1}{U(3^k)}\left(\frac{1}{c_k - 1} - \frac{1}{(c_k - 1)(c_k + 1)}\right)$$

$$= \frac{U(3^k - 1)}{U(3^k)} + \frac{1}{U(3^k)} \frac{c_k}{c_k^2 - 1}$$

$$= \frac{U(3^k - 1)(V(3^k)^2 - 1) + V(3^k)}{U(3^k)(V(3^k)^2 - 1)} \quad (4)$$

Now, using the fact that

$$U(3n) = U(n)(V(n)^2 - 1)$$

and

$$U(3n - 1) = U(n - 1)(V(n)^2 - 1) + V(n),$$

we see that the right side of (4) equals

$$\frac{U(3^{k+1} - 1)}{U(3^{k+1})}$$

which completes the induction step.

Equation (3) gives us an alternative proof of our Theorem above. By letting $k \to \infty$, we see that

$$\{c_0 - 1,\ c_0 + 1,\ c_1 - 1,\ c_1 + 1,\ \ldots\} = \lim_{k \to \infty} \frac{U(3^k - 1)}{U(3^k)} = \frac{1}{\beta} = \alpha.$$

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