BINET'S FORMULA FOR THE RECURSIVE SEQUENCE OF ORDER K

W. R. SPICKERMAN and R. N. JOYNER
East Carolina University, Greenville, NC 27834
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1. INTRODUCTION

The terms of a recursive sequence are usually defined by a recurrence procedure; that is, any term is the sum of preceding terms. Such a definition might not be entirely satisfactory, because the computation of any term could require the computation of all of its predecessors. An alternative definition gives any term of a recursive sequence as a function of the index of the term. Binet's formulas for the two simplest nontrivial recursive sequences are known. For the recursive sequence of order 2, the Fibonacci sequence, the formula

$$u_n = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1})$$

defines any Fibonacci number as a function of n and two constants α and β [1]. Similarly, for the recursive sequence of order 3, the Tribonacci sequence, the formula

$$u_n = \frac{\rho^2}{\rho^2 - 2\rho r \cos\theta + r^2}\rho^n + \frac{r(r - 2\rho\cos\theta)}{\rho^2 - 2\rho r\cos\theta + r^2} r^n \cos n\theta + \frac{r^2\cos\theta - \rho r(1 - 2\sin^2\theta)}{\sin\theta(\rho^2 - 2\rho r\cos\theta + r^2)} r^n \sin n\theta$$

defines any Tribonacci number as a function of n and three constants ρ, r, and θ [2].

In this paper, an analog of Binet's formula for the recursive sequence of order k ($k \geqslant 3$) is derived. The recursive sequence of order k is defined as follows:

$$u_n = 1 \qquad n = 0$$

$$u_n = \sum_{i=0}^{n-1} u_i \qquad 1 \leqslant n \leqslant k - 1$$

$$u_n = \sum_{i=n-k}^{n-1} u_i \qquad n \geqslant k.$$

The analog of Binet's formula defines any term of the recursive sequence of order k as a function of the index of the term and k constants.

2. BINET'S FORMULA FOR THE RECURSIVE SEQUENCE OF ORDER k

Binet's formula for the recursive sequence of order k is derived by solving the system of difference equations:

$$u_0 = 1$$

$$u_n = 2^{n-1} \qquad 1 \leqslant n \leqslant k - 1$$

$$u_{n+1} = \sum_{n-k+1}^{n} u_i \qquad n \geqslant k - 1.$$

Let $f(x) = \sum_{i=0}^{\infty} u_i x^i$ be the generating function for the solution. Then,

$$\left(1 - \sum_{j=1}^{k} x^j\right) f(x) = 1$$

or

$$f(x) = \frac{1}{1 - \sum_{j=1}^{k} x^j} = \frac{1}{\prod_{j=0}^{k-1}(1 - \alpha_j x)} = \frac{1}{p_k(x)},$$

where $1/\alpha$ is a zero of $1 - \sum_{j=1}^{k} x^j = 0$. Miller (see [3]) proved that the zeros of $p_k(1/x)$ are simple, consequently the roots of $p_k(x)$ are simple. Hence, $f(x)$ may be expressed by partial fractions as

$$f(x) = \frac{1}{\prod_{j=0}^{k-1}(1 - \alpha_j x)} = \sum_{j=0}^{k-1} \frac{A_j}{1 - \alpha_j x}, \text{ where } A_j = \frac{1}{\prod_{\substack{m=0\\m \neq j}}^{k-1}\left[1 - \alpha_m\left(\frac{1}{\alpha_j}\right)\right]}.$$

Further, since $\dfrac{1}{\prod_{\substack{m=0\\m \neq j}}^{k-1}\left[1 - \alpha_m\left(\frac{1}{\alpha_j}\right)\right]} = \dfrac{-\alpha_j}{p_k'\left(\frac{1}{\alpha_j}\right)}$, it follows that $A_j = \dfrac{-\alpha_j}{p_k'\left(\frac{1}{\alpha_j}\right)}$. Hence,

$$f(x) = \sum_{j=0}^{k-1} \frac{-\alpha_j}{p_k'\left(\frac{1}{\alpha_j}\right)} \cdot \frac{1}{1 - \alpha_j x} = \sum_{i=0}^{\infty}\left[\sum_{j=0}^{k-1} \frac{-\alpha_j}{p_k'\left(\frac{1}{\alpha_j}\right)}\right]\alpha_j^i x^i.$$

Therefore, $u_n = \sum_{j=0}^{k-1} \dfrac{-\alpha_j(\alpha_j)^n}{p_k'\left(\frac{1}{\alpha_j}\right)}$. Since $p_k'\left(\frac{1}{\alpha_j}\right) = -\sum_{m=1}^{k} m\left(\frac{1}{\alpha_j}\right)^{m-1}$ and

$$\left[1 - \sum_{m=1}^{k}\left(\frac{1}{\alpha_j}\right)^m\right] = 0 \text{ for } 0 \leqslant j \leqslant k - 1,$$

then $-\left(1 - \frac{1}{\alpha_j}\right)p_k'\left(\frac{1}{\alpha_j}\right) = 2 - (k + 1)\left(\frac{1}{\alpha_j}\right)^k$, and it follows that

$$u_n = \sum_{j=0}^{k-1} \frac{\alpha_j(\alpha_j)^n\left(1 - \frac{1}{\alpha_j}\right)}{2 - (k + 1)\left(\frac{1}{\alpha_j}\right)^k}.$$

Multiplying by $(\alpha_j/\alpha_j)^k$, yields $u_n = \sum_{j=0}^{k-1} \dfrac{\left(\alpha_j^{k+1} - \alpha_j^k\right)\alpha_j^n}{2\alpha_j^k - (k + 1)}$.

Let

$$r_k(x) = x^k - \sum_{j=0}^{k-1} x^j.$$

Miller [3] showed that $r_k(x) = 0$ has one real zero λ such that $1 < \lambda < 2$, with the remaining zeros inside the unit circle of the complex plane. Considering

$$q_k(x) = (x - 1) \cdot r_k(x) = 0$$

and using Descartes' Rule of Signs, it follows that $r_k(x) = 0$ has exactly one positive real zero when k is odd and that $r_k(x) = 0$ has exactly one positive real zero and exactly one negative real zero when k is even. Therefore, all other zeros of $r_k(x) = 0$ are complex and appear in conjugate pairs.

Now let

$$t = \left[\!\left[\frac{k-3}{2}\right]\!\right],$$

where $[\![\ ]\!]$ denotes the greatest integer function. Further, let α_j be a complex zero of $r_k(x) = 0$ if

$$0 \leqslant j \leqslant 2t + 1$$

and α_j be a real zero of $r_k(x) = 0$ if

$$2t + 2 \leqslant j \leqslant k - 1.$$

Also order the subscripts of the zeros so that $\alpha_m = \bar{\alpha}_j$ for

$$m = t + j + 1,\ 0 \leqslant j \leqslant t.$$

Consequently,

$$u_n = \sum_{j=0}^{t} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)} + \sum_{j=t+1}^{2t+1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)} + \sum_{j=2t+2}^{k-1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)}$$

$$= \sum_{j=0}^{t} \left[\frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)} + \frac{(\bar{\alpha}_j^{k+1} - \bar{\alpha}_j^k)\bar{\alpha}_j^n}{2\bar{\alpha}_j^k - (k+1)}\right] + \sum_{j=2t+2}^{k-1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)}$$

$$= \sum_{j=0}^{t} \frac{[2\bar{\alpha}_j^k - (k+1)](\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n + [2\alpha_j^k - (k+1)](\bar{\alpha}_j^{k+1} - \bar{\alpha}_j^k)\bar{\alpha}_j^n}{|2\alpha_j^k - (k+1)|^2}$$

$$+ \sum_{j=2t+2}^{k-1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)}$$

$$= \sum_{j=0}^{t} \frac{[2|\alpha_j|^{2k}\alpha_j - 2|\alpha_j|^{2k} - (k+1)\alpha_j^{k+1} + (k+1)\alpha_j^k]\alpha_j^n}{|2\alpha_j^k - (k+1)|^2}$$

$$+ \sum_{j=0}^{t} \frac{[2|\alpha_j|^{2k}\bar{\alpha}_j - 2|\alpha_j|^{2k} - (k+1)\bar{\alpha}_j^{k+1} + (k+1)\bar{\alpha}_j^k]\bar{\alpha}_j^n}{|2\alpha_j^k - (k+1)|^2}$$

$$+ \sum_{j=2t+2}^{k-1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)}$$

Applying Euler's formula,

$$\alpha_j = r_j(\cos\theta_j + i\sin\theta_j),\ \bar{\alpha}_j = r_j(\cos\theta_j - i\sin\theta_j),$$

the relation $|\alpha_j|^2 = r_j^2$, and simplifying yields

$$u_n = \sum_{j=0}^{t} r_j^n[A(k, j)\cos n\,\theta_j + B(k, j)\sin n\,\theta_j] + \sum_{j=2t+2}^{k-1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)},$$

where

$$A(k, j) = \frac{2r_j^k[2r_j^{k+1}\cos\theta_j - 2r_j^k - (k+1)r_j\cos(k+1)\theta_j + (k+1)\cos k\,\theta_j]}{4r_j^{2k} - 4(k+1)r_j^k\cos k\,\theta_j + (k+1)^2}$$

and

$$B(k, j) = \frac{-2r_j^k[2r_j^{k+1}\sin\theta_j - (k+1)r_j\sin(k+1)\theta_j + (k+1)\sin k\,\theta_j]}{4r_j^{2k} - 4(k+1)r_j^k\cos k\,\theta_j + (k+1)^2}$$

A form more suitable for computation of a single term is:

$$u_n = \sum_{j=0}^{t} \frac{2r_j^{n+k}[2r_j^{k+1}\cos(n+1)\theta_j - (k+1)r_j\cos(n+k+1)\theta_j + (k+1)\cos(n+k)\theta_j - 2r_j^k\cos n\,\theta_j]}{4r_j^{2k} - 4(k+1)r_j^k\cos k\,\theta_j + (k+1)^2} + \sum_{j=2t+2}^{k-1} \frac{(\alpha_j^{k+1} - \alpha_j^k)\alpha_j^n}{2\alpha_j^k - (k+1)}.$$

3. SOME NUMERICAL RESULTS

Let

$$C(k, j) = \frac{\alpha_j^{k+1} - \alpha_j^k}{2\alpha_j^k - (k+1)} \quad \text{for } (2t+2) \leqslant j \leqslant (k-1).$$

Then approximate values for the constants in the Binet formula

$$u_n = \sum_{j=0}^{t} r_j^n[A(k, j)\cos n\,\theta_j + B(k, j)\sin n\,\theta_j] + \sum_{j=2t+2}^{k-1} C(k, j)\alpha_j^n$$

for $2 \leqslant k \leqslant 10$ are given in the following table:

CONSTANTS IN BINET'S FORMULA FOR $2 \leqslant k \leqslant 10$

k	j	m	α_j, $\bar{\alpha}_j$	r_j	θ_j	$A(k, j)$	$B(k, j)$	$C(k, j)$
2	0	–	1.6180	–	–	–	–	0.7236
	1	–	-0.6180					0.2764
3	0	1	$-0.4196 \pm 0.6063i$	0.7374	2.1762	0.3816	0.0374	–
	2	–	1.8393	–	–	–	–	0.6184
4	0	1	$-0.0764 \pm 0.8147i$	0.8183	1.6643	0.2842	0.0563	–
	2	–	-0.7748	–	–	–	–	0.1495
	3	–	1.9276	–	–	–	–	0.5663

CONSTANTS IN BINET'S FORMULA FOR $2 \leqslant k \leqslant 10$ (continued)

k	j	m	α_j, $\bar{\alpha}_j$	r_j	θ_j	$A(k, j)$	$B(k, j)$	$C(k, j)$
5	0	2	$-0.6784 \pm 0.4585i$	0.8188	2.5472	0.2421	0.0178	-
	1	3	$0.1954 \pm 0.8489i$	0.8710	1.3446	0.2200	0.0654	-
	4	-	1.9659	-	-	-	-	0.5379
6	0	2	$-0.4619 \pm 0.7191i$	0.8547	2.1418	0.2012	0.0279	-
	1	3	$0.3903 \pm 0.8179i$	0.9062	1.1255	0.1741	0.0689	-
	4	-	-0.8403	-	-	-	-	0.1029
	5	-	1.9835	-	-	-	-	0.5218
7	0	3	$-0.7842 \pm 0.3600i$	0.8629	2.7112	0.1765	0.0103	-
	1	4	$-0.2407 \pm 0.8492i$	0.8826	1.8469	0.1703	0.0340	-
	2	5	$0.5289 \pm 0.7653i$	0.9303	0.9661	0.1398	0.0691	-
	6	-	1.9920	-	-	-	-	0.5125
8	0	3	$-0.6416 \pm 0.6064i$	0.8828	2.3844	0.1550	0.0168	-
	1	4	$0.6287 \pm 0.7085i$	0.9472	0.8450	0.1132	0.0672	-
	2	5	$-0.0469 \pm 0.9030i$	0.9042	1.6227	0.1461	0.0377	-
	6	-	-0.8763	-	-	-	-	0.0785
	7	-	1.9960	-	-	-	-	0.5071
9	0	4	$-0.8397 \pm 0.2948i$	0.8900	2.8040	0.1401	0.0067	-
	1	5	$0.1143 \pm 0.9140i$	0.9211	1.4464	0.1266	0.0398	-
	2	6	$0.7019 \pm 0.6539i$	0.9593	0.7500	0.0924	0.0641	-
	3	7	$-0.4755 \pm 0.7637i$	0.8996	2.1276	0.1368	0.0212	-
	8	-	1.9980	-	-	-	-	0.5040
10	0	4	$0.2462 \pm 0.9013i$	0.9344	1.3041	0.1106	0.0408	-
	1	5	$-0.3130 \pm 0.8584i$	0.9137	1.9205	0.1218	0.0242	-
	2	6	$0.7567 \pm 0.6039i$	0.9682	0.6735	0.0759	0.0602	-
	3	7	$-0.7399 \pm 0.5168i$	0.9025	2.5319	0.1259	0.0113	-
	8	-	-0.8990	-	-	-	-	0.0635
	9	-	1.9990	-	-	-	-	0.5022

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