GENERALIZED PELL POLYNOMIALS AND OTHER POLYNOMIALS

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1. INTRODUCTION

Following some of the techniques in [1] and [2], Walton [8] and [9] discussed several properties of the polynomial sequence $\{A_n(x)\}$ defined by the second-order recurrence relation

$$A_{n+2}(x) = 2xA_{n+1}(x) + A_n(x), \quad A_0(x) = q, \quad A_1(x) = p. \tag{1.1}$$

The first few terms of $\{A_n(x)\}$ are:

$$\begin{cases} A_0(x) = q, \; A_1(x) = p, \; A_2(x) = 2px + q, \; A_3(x) = 4px^2 + 2qx + p, \\ A_4(x) = 8px^3 + 4qx^2 + 4px + q, \; A_5(x) = 16px^4 + 8qx^3 + 12px^2 + 4qx + p, \end{cases} \tag{1.2}$$

. . . .

Using standard techniques, we easily obtain the Binet form

$$A_n(x) = \frac{(p - q\beta)\alpha^n - (p - q\alpha)\beta^n}{\alpha - \beta}, \tag{1.3}$$

where

$$\begin{cases} \alpha = x + \sqrt{x^2 + 1} \\ \beta = x - \sqrt{x^2 + 1} \end{cases} \tag{1.4}$$

are the roots of

$$t^2 - 2xt - 1 = 0 \tag{1.5}$$

so that

$$\alpha + \beta = 2x, \quad \alpha - \beta = 2\sqrt{x^2 + 1}, \quad \alpha\beta = -1. \tag{1.6}$$

In this paper we relate part of the work in [8] and [9] to other well-known polynomials. Thus, only some basic features of $\{A_n(x)\}$ will be examined.

It should be noted in passing that the expression for $\{A_n(x)\}$ in (1.3) is in agreement with the form for the n^{th} term of more general sequences of polynomials considered in [6]. Properties of the general sequence of numbers $\{W_n\}$ given in [4] are also readily generalized to yield properties of $\{A_n(x)\}$.

Note that when $x = 1/2$ in (1.1) we obtain the generalized Fibonacci number sequence $\{H_n\}$ whose basic properties are described in [3]. Furthermore, if we also let $p = 1$, $q = 0$ in (1.1), then we derive the sequence $\{F_n\}$ of Fibonacci numbers. Letting $p = 1$, $q = 2$ in (1.1) with $x = 1/2$, we obtain the sequence $\{L_n\}$ of Lucas numbers.

For unspecified x, the Pell polynomials $P_n(x)$ occur when $p = 1$ and $q = 0$ in (1.1), while for $p = 2x$ and $q = 2$ the Pell-Lucas polynomials $Q_n(x)$ arise. Relationships among $P_n(x)$ and $Q_n(x)$ are developed in [5]. Hence, polynomials of the sequence $\{A_n(x)\}$ may be called *generalized Pell polynomials*.

Readers may find some interest in specializing the results for $\{A_n(x)\}$ to the polynomial sequences $\{P_n(x)\}$ and $\{Q_n(x)\}$, and to the number sequences $\{H_n\}$, $\{F_n\}$, and $\{L_n\}$. Some of the specialized formulas for $\{H_n\}$ are, in fact, supplied in [8] and [9].

Though it is not strictly pertinent to this article, we wish to record an important formula for $\{A_n(x)\}$ which was not included in [9], namely, *Simson's formula*:

$$A_n^2(x) - A_{n+1}(x)A_{n-1}(x) = (-1)^n(q^2 - p^2 + 2px). \tag{1.7}$$

2. $A_n(x)$ AND CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

In [8] and [9] it is shown that

$$A_n(x) = q\sum_{m=0}^{[n/2]}\binom{n-m}{m}(2x)^{n-2m} + (p - 2qx)\sum_{m=0}^{\left[\frac{n-1}{2}\right]}\binom{n-1-m}{m}(2x)^{n-1-2m} \tag{2.1}$$

with $n \geqslant 1$. Furthermore, from [5] and [7], we have, respectively, the Pell polynomials given by

$$P_n(x) = \sum_{m=0}^{\left[\frac{n-1}{2}\right]}\binom{n-m-1}{m}(2x)^{n-2m-1} \tag{2.2}$$

and the Chebyshev polynomials of the second kind given by

$$U_n(x) = \sum_{m=0}^{[n/2]}(-1)^m\binom{n-m}{m}(2x)^{n-2m}. \tag{2.3}$$

Letting x be replaced by ix in (2.3), we see that

$$\sum_{m=0}^{[n/2]}\binom{n-m}{m}(2x)^{n-2m} = (-i)^n U_n(ix) = P_{n+1}(x), \tag{2.4}$$

so that (2.1) can be rewritten as

$$\begin{aligned} A_n(x) &= q(-i)^n U_n(ix) + (p - 2qx)(-i)^{n-1}U_{n-1}(ix) \\ &= qP_{n+1}(x) + (p - 2qx)P_n(x) \\ &= pP_n(x) + qP_{n-1}(x), \end{aligned} \tag{2.5}$$

which is another form of (1.1), which could also have been obtained by using the generating functions for $A_n(x)$ (given in [9]) and $P_n(x)$ (given in [5]) or their respective Binet forms.

3. HYPERBOLIC FUNCTIONS AND $A_n(x)$

Elementary methods enable us to derive, when $x = \sinh w = (e^w - e^{-w})/2$,

$$A_{2k}(x) = \{p \sinh 2kw + q \cosh(2k - 1)w\}/\cosh w \tag{3.1}$$

and

$$A_{2k+1}(x) = \{p \cosh(2k+1)w + q \sinh 2kw\}/\cosh w. \tag{3.2}$$

To achieve these results, we use the Binet form (1.3) and

$$\alpha = e^{w},\ \beta = -e^{-w},\ \alpha - \beta = 2 \cosh w = e^{w} + e^{-w}.$$

If we now use formulas (6.1) and (6.2) of [5], then (3.1) and (3.2) become (2.5) for the cases $n = 2k$ and $n = 2k + 1$, respectively.

4. GEGENBAUER POLYNOMIALS AND $A_n(x)$

The Gegenbauer polynomials C_n^k for $k > -\frac{1}{2}$, $k \neq 0$, are given in [7] by

$$C_n^k(x) = \frac{1}{\Gamma(k)} \sum_{m=0}^{[n/2]} (-1)^m \frac{\Gamma(n-m+k)}{\Gamma(n-m+1)} \binom{n-m}{m} (2x)^{n-2m}, \tag{4.1}$$

where $\Gamma(x)$ is the Gamma function. With $k = 1$, we have

$$C_n^1(x) = \sum_{m=0}^{[n/2]} (-1)^m \binom{n-m}{m} (2x)^{n-2m} = U_n(x), \tag{4.2}$$

so that by (2.5) we obtain

$$A_n(x) = q(-i)^n C_n^1(ix) + (p - 2qx)(-i)^{n-1} C_{n-1}^1(ix). \tag{4.3}$$

5. DETERMINANTAL GENERATION OF $A_n(x)$

Let us define two functional determinants $\Delta_{n-1}(x)$ and $\delta_{n-1}(x)$ of order $n-1$ as follows, where d_{ij} denotes the element in the i^{th} row and j^{th} column:

$$\Delta_{n-1}(x): \begin{cases} d_{ii} = 2px + q & i = 1, 2, \ldots, n-1 \\ d_{i,i+1} = p & i = 1, 2, \ldots, n-2 \\ d_{i,i-1} = -1 & i = 2, 3, \ldots, n-1 \\ d_{ij} = 0 & \text{otherwise} \end{cases} \tag{5.1}$$

$$\delta_{n-1}(x): \text{ as for } \Delta_{n-1}(x) \text{ except that } d_{i,i+1} = -p,\ d_{i,i-1} = 1. \tag{5.2}$$

Expansion along the first row then yields:

$$\begin{aligned} \Delta_{n-1}(x) &= (2px + q)\Delta_{n-2}(x) + p\Delta_{n-3}(x) && (5.3) \\ &= p\{2xP_{n-1}(x) + P_{n-2}(x)\} + qP_{n-1}(x) && \text{by (5.5) of [5]} \\ &= pP_n(x) + qP_{n-1}(x) && \text{by (1.1) of [5]} \\ &= A_n(x) && \text{by (2.5).} \end{aligned}$$

Similarly,

$$\delta_{n-1}(x) = A_n(x). \tag{5.4}$$

As mentioned at the end of §2, a generating function for $A_n(x)$ is given in [9].

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