ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-381 *Proposed by Dejan M. Petković, Niš, Yugoslavia*

Let N be the set of all natural numbers and let $m \in N$. Show that

$$\text{(i)}\quad \zeta(2m-2) = \frac{(-)^m \bar{u}^{2m-2}(m-1)}{(2m-1)!} + \sum_{i=2}^{m-1} \frac{(-)^i \bar{u}^{2i-2}}{(2i-1)!}\cdot \zeta(2m-2i),\ m \geqslant 2,$$

$$\text{(ii)}\quad \beta(2m-1) = \sum_{i=1}^{m-1} \frac{(-)^i \bar{u}^{2i}}{2^{2i}(2i)!}\cdot \beta(2m-2i-1),\ m \geqslant 2,$$

$$\text{(iii)}\quad \zeta(2m) = \frac{2^{2m}}{2^{2m}-1} \sum_{i=0}^{m-1} \frac{(-)^i \bar{u}^{2i+1}}{2^{2i+1}(2i+1)!}\cdot \beta(2m-2i-1),\ m \geqslant 1,$$

where

$$\zeta(m) = \sum_{n=1}^{\infty} n^{-m},\ m \geqslant 2, \text{ are Riemann zeta numbers}$$

and

$$\beta(m) = \sum_{n=1}^{\infty} (-)^{n-1}(2n-1)^{-m},\ m \geqslant 1.$$

H-382 *Proposed by Andreas N. Philippou, Patras, Greece*

For each fixed positive integer k, define the sequence of polynomials $A_{n+1}^{(k)}(p)$ by

$$A_{n+1}^{(k)}(p) = \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}\left(\frac{1-p}{p}\right)^{n_1 + \cdots + n_k} \quad (n \geqslant 0,\ -\infty < p < \infty), \tag{1}$$

where the summation is taken over all nonnegative integers $n_1, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n + 1$. Show that

$$A_{n+1}^{(k)}(p) \leqslant (1-p)p^{-(n+1)}(1-p^k)^{[n/k]} \quad (n \geqslant k-1,\ 0 < p < 1), \tag{2}$$

where $[n/k]$ denotes the greatest integer in (n/k).

It may be noted that (2) reduces to

$$F_n^{(k)} \leqslant 2^n \left(\frac{2^k - 1}{2^k}\right)^{[n/k]} \quad (n \geqslant k-1) \tag{3}$$

and

$$F_n \leqslant 2^n(3/4)^{[n/2]} \quad (n \geqslant 1), \tag{4}$$

where $\{F_n^{(k)}\}_{n=0}^{\infty}$ and $\{F_n\}_{n=0}^{\infty}$ denote the Fibonacci sequence of order k and the usual Fibonacci sequence, respectively, if $p = 1/2$ and $p = 1/2$, $k = 2$.

References

1. J. A. Fuchs. Problem B-39. *The Fibonacci Quarterly* **2**, no. 2 (1964):154.
2. A. N. Philippou. Problem H-322. *The Fibonacci Quarterly* **19**, no. 1 (1981): 93.

H-383 *Proposed by Clark Kimberling, Evansville, IN*

For any $x > 0$, let

$$c_1 = 1, \quad c_2 = x, \quad \text{and} \quad c_n = \frac{1}{n}\sum_{i=1}^{n} c_i c_{n-i} \quad \text{for } n = 3, 4, \ldots .$$

Prove or disprove that there exists $y > 0$ such that $\lim\limits_{n\to\infty} y^n c_n = 1$.

H-384 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany*

Show that for $n = 0, 1, 2, \ldots,$

$$\sum_{k=0}^{\infty}\frac{1}{(2k)!}\prod_{j=0}^{k-1}\left[\left(n + \frac{1}{2}\right)^2 - j^2\right] = \frac{\sqrt{5}}{2}F_{2n+1}$$

SOLUTIONS

Waiting Again

H-358 *Proposed by Andreas N. Philippou, University of Patras, Greece (Vol. 21, no. 3, August, 1983)*

For any fixed integers $k \geqslant 1$ and $r \geqslant 1$, set

$$f_{n+1,r}^{(k)} = \sum_{n_1,\ldots,n_k}\binom{n_1 + \cdots + n_k + r - 1}{n_1, \ \ldots \ , n_k \ , r - 1}, \ n \geqslant 0,$$

where the summation is over all nonnegative integers $n_1, \ldots, n_k$ satisfying the relation $n_1 + 2n_2 + \cdots + kn_k = n$. Show that

$$\sum_{n=0}^{\infty}(f_{n+1,r}^{(k)}/2^n) = 2^{rk}.$$

You may note that the present problem reduces to H-322(c) for $r = 1$ (and $k \geqslant 2$), because of Theorem 2.1 of Philippou and Muwafi [1]. In addition, the present problem includes as special cases [for $k = 1$, $r = 1$, and $k = 1$, r ($\geqslant 1$)] the following infinite sums; namely,

$$\sum_{n=0}^{\infty}(1/2^n) = 2 \quad \text{and} \quad \sum_{n=0}^{\infty}\left[\binom{n + r - 1}{n}/2^n\right] = 2^r.$$

Reference

1. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order K." *The Fibonacci Quarterly* **20**, no. 1 (1982):28-32.

Solution by the proposer

Set

$$f^{(k)}_{n+1,\,r}(p) = \sum_{\substack{n_1,\,\ldots,\,n_k \ni \\ n_1,\,2n_2+\cdots+kn_k=n}} \binom{n_1+\cdots+n_k+r-1}{n_1,\;\ldots,\;n_k,\;\;r-1} p^n \left(\frac{1-p}{p}\right)^{n_1+\cdots+n_k} \tag{1}$$

$$(n \geqslant 0,\; -\infty < p < \infty).$$

It follows, by means of the transformation $n_i = m_i$ $(1 \leqslant i \leqslant k)$ and

$$n = m + \sum_{i=1}^{k} (i-1)m_i,$$

that

$$\sum_{n=0}^{\infty} f^{(k)}_{n+1,\,r}(p)$$

$$= \sum_{n=0}^{\infty} \sum_{\substack{n_1,\,\ldots,\,n_k \ni \\ n_1,\,2n_2+\cdots+kn_k=n}} \binom{n_1+\cdots+n_k+r-1}{n_1+\cdots+n_k}\binom{n_1+\cdots+n_k}{n_1,\;\ldots,\;n_k} p^n \left(\frac{1-p}{p}\right)^{n_1+\cdots+n_k}$$

$$= \sum_{m=0}^{\infty} \binom{m+r-1}{m}\left(\frac{1-p}{p}\right)^m \sum_{\substack{m_1,\,\ldots,\,m_k \ni \\ m_1+\cdots+m_k=m}} \binom{m}{m_1,\;\ldots,\;m_k} p^{m_1+2m_2+\cdots+km_k}$$

$$= \sum_{m=0}^{\infty} \binom{m+r-1}{m}\left(\frac{1-p}{p}\right)^m (p+p^2+\cdots+p^k)^m, \text{ by the multinomial theorem,}$$

$$= \sum_{m=0}^{\infty} (-1)^m \binom{-r}{m}(1-p^k)^m = (1-(1-p^k))^{-r}, \text{ for } |1-p^k| < 1,$$

by the binomial theorem,

$$= p^{-kr}, \text{ for } k \text{ odd and } 0 < p < \sqrt[k]{2}, \text{ or } k \text{ even and } -\sqrt[k]{2} < p < \sqrt[k]{2}. \tag{2}$$

For $p = 1/2$, (1) and (2) establish the problem. For $r = 1$, (1) and (2) show H-348.

Also solved by Paul S. Bruckman.

Zetanacci

H-359 *Proposed by Paul S. Bruckman, Carmichael, CA*
(Vol. 21, no. 3, August 1983)

Define the "Zetanacci" numbers $Z(n)$ as follows:

$$Z(n) = \prod_{p^e \| n} F_{e+1}, \; n = 1, 2, 3, \ldots \text{ [with } Z(1) = 1]. \tag{1}$$

For example, $Z(n) = 1$, $n = 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, \ldots$; $Z(n) = 2$, $n = 4, 9, 12, 18, 20, \ldots$; $Z(8) = 3$, $Z(16) = 5$), $Z(135{,}000) = Z(2^3 3^3 5^4) = 45$, and so forth.

(A) Show that the (Dirichlet) generating function of the Zetanacci numbers is given by:

$$\sum_{n=1}^{\infty} Z(n)n^{-s} = \prod_{p} (1 - p^{-s} - p^{-2s})^{-1},$$

(B) Show that

$$\prod_p (1 - p^{-s} - p^{-2s}) = \sum_{n=1}^{\infty} \mu(P(n)) \cdot |\mu(n/P(n))| \cdot n^{-s},$$

where μ is the Möbius function and

$$P(n) = \prod_{p|n} p \text{ [with } P(1) = 1].$$

Solution by C. Georghiou, University of Patras, Greece

The solution of the problem is based on the following known proposition [see, e.g., G. Polya & G. Szego, *Problems and Theorems in Analysis II* (Springer-Verlag, 1976), pp. 121, 312]:

"Let $f(n)$ be a multiplicative arithmetical function (m.a.f.). Then we have

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p (1 + f(p)p^{-s} + f(p^2)p^{-2s} + f(p^3)p^{-3s} + \cdots) \qquad (*)$$

and conversely, if (*) holds, then $f(n)$ is a m.a.f."

(A) From the definition, we note that $Z(n)$ is a m.a.f. and $Z(p^k) = F_{k+1}$ for every prime p. Therefore, from (*), we have

$$\sum_{n=1}^{\infty} Z(n)n^{-s} = \prod_p (1 + F_2p^{-s} + F_3p^{-2s} + F_4p^{-3s} + \cdots)$$

$$= \prod_p (1 - p^{-s} - p^{-2s})^{-1},$$

where we used the fact that the (ordinary) generating function of the sequence $\{F_{n+1}\}_{n=0}^{\infty}$ is $f(x) = (1 - x - x^2)^{-1}$.

(B) We have, according to (*),

$$\prod_p (1 - p^{-s}p^{-2s}) = \prod_p (1 + f(p)p^{-s} + f(p^2)p^{-2s} + f(p^3)p^{-3s} + \cdots)$$

$$= \sum_{n=1}^{\infty} f(n)n^{-s},$$

where $f(n)$ is a m.a.f. and $f(1) = 1$, $f(p) = -1$, $f(p^2) = -1$, and $f(p^k) = 0$ for every prime p and $k > 2$. Thus the problem reduces to that of finding a m.a.f. $f(n)$ with the above-stated properties. By choosing $f(n)$ such that $f(1) = 1$ and

$$f(p^k) = \mu(p) \cdot |\mu(p^{k-1})|,$$

where μ is the Mobius function, for every prime p and $k \geqslant 1$ the above requirements are satisfied. If $n = p_{m_1}^{n_1} p_{m_2}^{n_2} \ldots p_{m_k}^{n_k}$, then since μ is a m.a.f., we have

$$f(n) = \mu(p_{m_1}, p_{m_2}, \ldots, p_{m_k}) \cdot |\mu(n/(p_{m_1}p_{m_2} \ldots p_{m_k}))|$$

$$= \mu(p(n) \cdot |\mu(n/P(n))|$$

from the definition of $P(n)$, and this proves (B).

Also solved by L. Kuipers and the proposer.

Say A

H-360 *Proposed by M. Wachtel, Zurich, Switzerland*
(Vol. 21, no. 4, November, 1983)

Let:
$$F_nF_{n+1} + F_{n+2}^2 = A_1$$
$$F_{n+1}F_{n+2} + F_{n+3}^2 = A_2$$
$$F_{n+2}F_{n+3} + F_{n+4}^2 = A_3$$

Show that:

1. no integral divisor of A is congruent to 3 or 7 modulo 10,
2. $A_1A_2 + 1$, as well as $A_1A_3 + 1$, are products of two consecutive integers.

Solution by Paul S. Bruckman, Fair Oaks, CA

We make a change in notation. Let

$$B_n = F_nF_{n+1} + F_{n+2}^2 \tag{1}$$
$$C_n = B_nB_{n+1} + 1, \tag{2}$$
$$D_n = B_nB_{n+2} + 1,\ n = 0, 1, 2, \ldots . \tag{3}$$

Note that

$$B_n = F_nF_{n+1} + F_{n+3}F_{n+1} + (-1)^{n+1} = F_{n+1}(F_{n+3} + F_n) - (-1)^n$$
$$= F_{n+1}(F_{n+2} + F_{n+1} + F_{n+2} - F_{n+1}) - (-1)^n,$$

or

$$B_n = 2F_{n+1}F_{n+2} - (-1)^n. \tag{4}$$

Proof of Part 1: It is sufficient to prove that no prime p with $p \equiv \pm 3$ (mod 10) divides B_n (for all n), since any number congruent to 3 or 7 (mod 10) divisible by such a prime. Note that

$$B_n = F_nF_{n+1} + (F_{n+1} + F_n)^2 = F_{n+1}^2 + 3F_{n+1}F_n + F_n^2,$$

or upon factorization:

$$B_n = (F_{n+1} + \alpha^2F_n)(F_{n+1} + \beta^2F_n), \tag{5}$$

where α and β are the usual Fibonacci constants.

Suppose p is any prime with $p \equiv \pm 3$ (mod 10). Then, $(5/p) = (p/5) = -1$. According to the calculus of "complex residues" (see [1]), we can define

$$\alpha \equiv 2^{-1}(1 + \sqrt{5}) \quad \text{and} \quad \beta \equiv 2^{-1}(1 - \sqrt{5}) \pmod{p}$$

as "complex residues" and manipulate such quantities algebraically in a manner analogous to that employed with ordinary complex numbers. In this proof, we assume that all congruences are modulo p and omit the notation "(mod p)" where no confusion is likely to arise.

Assume $B_n \equiv 0$ (mod p). Then one of the two factors indicated in (5) must vanish (mod p). If $F_{n+1} + \alpha^2F_n \equiv 0$, then $\alpha^{n+1} - \beta^{n+1} + \alpha^{n+2} - \beta^{n-2} \equiv 0$, implying

$$\alpha^{n+1}(1 + \alpha) \equiv \beta^{n-2}(\beta^3 + 1) \Rightarrow \alpha^{n+3} \equiv 2\beta^n \Rightarrow \alpha^{2n+3} \equiv 2(-1)^n$$

and

$$\beta^{2n+3} \equiv -2^{-1}(-1)^n.$$

Hence,

$$F_{2n+3} = 5^{-\frac{1}{2}}(\alpha^{2n+3} - \beta^{2n+3}) \equiv (2 + 2^{-1})5^{-\frac{1}{2}}(-1)^n \equiv 2^{-1}5^{\frac{1}{2}}(-1)^n.$$

Similarly, if $F_{n+1} + \beta^2 F_n \equiv 0$, then $F_{2n+3} = -2^{-1}5^{\frac{1}{2}}(-1)^n$. Hence, $B_n \equiv 0$ implies $F_{2n+3} \equiv \pm 2^{-1}5^{\frac{1}{2}}$. However, this is impossible, since F_{2n+3} is "real," while $5^{\frac{1}{2}}$, and thus $\pm 2^{-1}5^{\frac{1}{2}}$ are "imaginary" (mod p). This contradiction establishes that $B_n \not\equiv 0$ (mod p), and hence the desired result.

Proof of Part 2: Using (2) and (4),

$$\begin{aligned} C_n &= (2F_{n+1}F_{n+2} - (-1)^n)(2F_{n+2}F_{n+3} + (-1)^n) + 1 \\ &= 4F_{n+1}F_{n+2}^2F_{n+3} - 2(-1)^n F_{n+2}(F_{n+3} - F_{n+1}) \\ &= 2F_{n+2}^2(2F_{n+1}F_{n+3} - (-1)^n) \\ &= 2F_{n+2}^2\{2(F_{n+2}^2 - (-1)^{n+1}) - (-1)^n\}, \end{aligned}$$

or

$$C_n = 2F_{n+2}^2(2F_{n+2}^2 + (-1)^n). \tag{6}$$

Also,

$$\begin{aligned} D_n &= (2F_{n+1}F_{n+2} - (-1)^n)(2F_{n+3}F_{n+4} - (-1)^n) + 1 \\ &= 4F_{n+1}F_{n+2}F_{n+3}F_{n+4} - 2(-1)^n(F_{n+1}F_{n+2} + F_{n+3}F_{n+4}) + 2 \\ &= 4F_{n+2}F_{n+3}(F_{n+3} - F_{n+2})(F_{n+3} + F_{n+2}) \\ &\quad - 2(-1)^n\{F_{n+2}(F_{n+3} - F_{n+2}) + F_{n+3}(F_{n+3} + F_{n+2})\} + 2 \\ &= 4F_{n+2}F_{n+3}(F_{n+3}^2 - F_{n+2}^2) - 2(-1)^n(2F_{n+2}F_{n+3} - F_{n+2}^2 + F_{n+3}^2) + 2 \\ &= (F_{n+3}^2 - F_{n+2}^2)(4F_{n+2}F_{n+3} - 2(-1)^n) - (-1)^n(4F_{n+2}F_{n+3} - 2(-1)^n) \\ &= (F_{n+3}^2 - F_{n+2}^2 - (-1)^n)(4F_{n+2}F_{n+3} - 2(-1)^n) \\ &= 2(F_{n+3}^2 - F_{n+3}F_{n+1})(2F_{n+2}F_{n+3} - (-1)^n) \\ &= 2F_{n+3}(F_{n+3} - F_{n+1})(2F_{n+2}F_{n+3} - (-1)^n), \end{aligned}$$

or

$$D_n = 2F_{n+2}F_{n+3}(2F_{n+2}F_{n+3} - (-1)^n). \tag{7}$$

We see from (6) and (7) that C_n and D_n are equal to products of two consecutive integers. Q.E.D.

Reference

1. Paul S. Bruckman. "Some Divisibility Properties of Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **17**, no. 1 (1979):42-49.

Also solved by L. Kuipers and the proposer.

Pell-Mell

H-361 *Proposed by Verner E. Hoggatt, Jr., deceased*
(Vol. 21, no. 4, November, 1983)

Let $H_n = P_{2n}/2$, $n > 0$, where P_n denotes the n^{th} Pell number. Show that

$$H_m + H_n = P_k$$
$$H_m + H_n = P_k + P_{k-1}$$

if and only if $m = n + 1$, where $k = 2n + 1$ and

$$P_{2n+2}/2 + P_{2n}/2 = ((2P_{2n+1} + P_{2n}) + P_{2n})/2 = P_{2n+1} + P_{2n}.$$

Editorial Note: Refer to the January 1972 article on the Generalized Zeckendorf Theorem for Pell Numbers.

Solution by Paul S. Bruckman, Fair Oaks, CA

We recall or indicate (without proof) some of the basic definitions and properties of the Pell and "modified Pell" numbers:

$$P_n \equiv \frac{1}{2\sqrt{2}}(\alpha^n - \beta^n); \quad Q_n \equiv \frac{1}{2}(\alpha^n + \beta^n),\ n = 0, 1, 2, \ldots, \tag{1}$$
where $\alpha \equiv 1 + \sqrt{2}$, $\beta \equiv 1 - \sqrt{2}$.

$$P_{n+2} = 2P_{n+1} + P_n; \quad Q_{n+2} = 2Q_{n+1} + Q_n. \tag{2}$$

P_n and Q_n are increasing with n, except for $Q_0 = Q_1 = 1$;
P_n and Q_n are positive, except for $P_0 = 0$. (3)

$P_u | P_v$ iff $u | v$; $Q_u | Q_v \Rightarrow u | v$. (4)
Setting $u = 2$, we see that P_n is even iff n is even.

$$Q_n^2 - 2P_n^2 = (-1)^n;\ \text{hence, } Q_n \text{ is odd for all } n. \tag{5}$$

$$P_{(a+1)b} + P_{(a-1)b} = 2P_b Q_{ab}; \quad Q_{(a+1)b} - Q_{(a-1)b} = 2Q_b Q_{ab},\ \text{if } b \text{ is odd.} \tag{6}$$

$$P_n + P_{n-1} = Q_n. \tag{7}$$

$$P_{2m} + P_{2n} = \begin{cases} 2P_{m+n}Q_{m-n}, & \text{if } m+n \text{ is even;} \\ 2P_{m-n}Q_{m+n}, & \text{if } m+n \text{ is odd.} \end{cases} \tag{8}$$

Most of these identities and properties follow readily from the definitions in (1), or are obtainable from the abundant literature on these sequences. Given two positive integers m and n, we define $s \equiv m + n$ and $d \equiv m - n$, where without loss of generality, we can assume $m \geqslant n$. We first note that there is an error in the statement of the problem; the first part of the problem should say:

$$H_m + H_n = P_k \text{ if and only if } m = n, \text{ in which case } k = 2n. \tag{9}$$

Proof of Part 1: The proposed equation is equivalent to the following:

$$P_{2m} + P_{2n} = 2P_k. \tag{10}$$

Hence, P_k is the arithmetic mean of P_{2m} and P_{2n}. Since the P_i's are increasing with i and since $m \geqslant n$, this implies: $2n \leqslant k \leqslant 2m$. We consider two possibilities: $m + n$ is even or $m + n$ is odd.

(a) s is even: Then, using (8), we must solve $P_k = P_s Q_d$. Thus, from (4), $s|k$, or $k = rs$ for some $r \geqslant 1$. Since $2n \leqslant r(m + n) \leqslant 2m$, we must have $r = 1$; hence, since $P_s > 0$, we must have $Q_d = 1$ and $d = 0$ or 1. Since d is even, $d = 0$, i.e., $m = n$, so $k = 2n$. This is the only solution of (10) in this case.

(b) s is odd: Again using (8), we are, therefore, required to solve $P_k = P_d Q_s$. Hence, again using (4), $d|k$, or $k = rd$ for some $r \geqslant 1$. If r is even, so is k; therefore, P_k [using (4)]. But d is odd; hence, P_d and Q_s are odd [by (4) and (5)], making it impossible for P_k to be even. This contradiction shows that r must be odd. Incidentally, this also shows that k must be odd. If $r = 1$, then (since $d \geqslant 1$) we have $Q_s = 1$ and $s = 0$ or 1, which is impossible, because $s \geqslant 3$. Therefore, r must be odd and greater than 2. Now the assumed equation implies

$$P_k = P_{rd} = P_d Q_s = 2P_d Q_{(r-1)d} - P_{(r-2)d},$$

using the first part of (6). Since $r > 2$ and $d \geqslant 1$,

$$P_{(r-2)d} > 0 \quad \text{and} \quad P_d \geqslant 1.$$

Hence, $P_d Q_s < 2P_d Q_{(r-1)d}$, which implies

$$Q_s < 2Q_{(r-1)d} < Q_{(r-1)d+1},$$

using (2). Then, by the property in (3), $s < (r - 1)d + 1$, or equivalently, $2m \leqslant k$. However, since $2n \leqslant k \leqslant 2m$, this implies that $k = 2m$, i.e., k is even: CONTRADICTION! Therefore, no solution of (10) exists in this case. This establishes (9).

Proof of Part 2: We see from (7) that the proposed equation is equivalent to

$$P_{2m} + P_{2n} = 2Q_k. \tag{11}$$

We again consider two cases: s is even or s is odd.

(a) s is even: Then, using (8), we are required to solve $Q_k = P_s Q_d$. Since s is even, so is P_s, hence Q_k. However, this is impossible, since Q_k is odd for all k. This contradiction eliminates any solutions in this case.

(b) s is odd: Now we are required to solve $Q_k = P_d Q_s$. Using (4), we have $s|k$, or $k = rs$ for some $r \geqslant 1$. If $r = 1$, then $Q_k = Q_s > 0$, so $P_d = 1$, implying that $k = 1$. Then, $m = n + 1$ and $k = 2n + 1$. This is a solution to equation (11). Suppose $r \geqslant 2$. Then, since $Q_{rs} - Q_{(r-2)s} = 2Q_s Q_{(r-1)s}$ [from (6)], we have

$$Q_k = Q_{rs} = P_d Q_s > 2Q_s Q_{(r-1)s},$$

implying that $P_d > 2Q_{(r-1)s}$. But clearly $2Q_n > P_n$ for all n [using (7)]. Thus, $P_d > P_{(r-1)s}$, which implies $d > (r - 1)s$, i.e., $(m - n) > (r - 1)(m + n)$. This can be true only if $r = 1$, which contradicts the hypothesis that $r \geqslant 2$.

Hence, $H_m + H_n = Q_k$ if and only if $m = n + 1$, where $k = 2n + 1$. Q.E.D.

Also solved by L. Kuipers.

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