ON TRIANGULAR FIBONACCI NUMBERS

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In Memory of Vern Hoggatt

Let F_n denote the n^{th} Fibonacci number:

$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}.$

Tallman [2] noted that $0 = F_0$, $1 = F_1 = F_2$, $3 = F_4$, $21 = F_8$, and $55 = F_{10}$ are triangular, i.e., of the form $k(k + 1)/2$, and asked if any more Fibonacci numbers are triangular. In this paper, we develop some congruences which must be satisfied by n if F_n is triangular. As a result, we prove that there are no more triangular numbers among the first billion Fibonacci numbers.

Moreover, the congruences developed here are so strikingly similar that they suggest an approach to proving that the known triangular Fibonacci numbers are in fact the only ones. A pattern is strongly suggested, but unfortunately any underlying generality remains elusive, leaving us with a good notion of how to test, but with no assurances that such tests will succeed. Thus, in a sense, the results in this paper constitute mere *number crunching*, albeit on a rather massive scale, given the simplicity of the techniques.

Throughout this paper, let

$A = 2^3 3 \cdot 5 = 120$

$B = 7A = 2^3 3 \cdot 5 \cdot 7 = 840$

$C = 6B = 2^4 3^2 5 \cdot 7 = 5040$

$D = 11C = 2^4 3^2 5 \cdot 7 \cdot 11 = 55{,}440$

$E = 10D = 2^5 3^2 5^2 7 \cdot 11 = 554{,}400$

$F = 13E = 2^5 3^2 5^2 7 \cdot 11 \cdot 13 = 7{,}207{,}200$

$G = 17F = 2^5 3^2 5^2 7 \cdot 11 \cdot 13 \cdot 17 = 122{,}522{,}400$

$H = 19G = 2^5 3^2 5^2 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 = 2{,}327{,}925{,}600$

Our approach is to show successively that

$$\text{if } F_n \text{ is triangular, then } n \equiv 0, 1, 2, 4, 8, 10, M/2 \text{ or } M - 1 \pmod{M} \tag{1}$$

for $M = A, \ldots, H$. Once (1) is established for $M = H$, it follows at once that there are no new triangular Fibonacci numbers with subscript less than one billion.

At the heart of what we do here is the simple observation that an integer f is triangular if and only if $8f + 1$ is a square.

If p is an odd prime, let $Z(p)$ be the entry point of p in the Fibonacci sequence. That is, $Z(p)$ is the subscript of the first Fibonacci number divisible by p. Then $p|F_n$ if and only if $Z(p)|n$. Tables of $Z(p)$ for $p < 10^4$ may be found in [1].

Further, let $k(p)$ be the period of the Fibonacci sequence modulo p. It is known that:

If $Z(p) = 2m + 1$, then $k(p) = 4Z(p)$.

If $Z(p) = 2(2m + 1)$, then $k(p) = Z(p)$.

If $Z(p) = 2^{a}(2m + 1)$ with $a \geqslant 2$, then $k(p) = 2Z(p)$.

What we will do is to find primes p for which $k(p)$ divides the new modulus but not the old one, and then eliminate most choices of n relative to the new modulus by showing that $1 + 8F_n$ is not a quadratic residue modulo p. The same thing may be done with composite moduli for alleged resedues, but it was necessary to do so only once.

Lemma. If F is triangular, then $n \equiv 0, 1, 2, 4, 8, 10, 20, 24$, or $39 \pmod{40}$.

Proof: We cannot have $n \equiv 3, 5, 6$, or $7 \pmod{10}$ or else $1 + 8F_n$ is a nonresidue (mod 11). We rule out $n \equiv 9, 11, 12, 14$, or $18 \pmod{20}$ to avoid having $1 + 8F_n$ be a nonresidue (mod 5). Similarly, we cannot have $n \equiv 3, 5$, or $6 \pmod{8}$ or else $1 + 8F_n$ is a nonresidue (mod 3). Finally, $n \equiv 28 \pmod{40}$ is impossible because $1 + 8F_{28}$ is a nonresidue (mod 41).

Theorem. (1) holds for $M = A, B, C, D, E, F, G$, and H.

Proof: The lemma and Table 1 establish the result for $M = A$; in Table 1 and the following tables, the entry gives a modulus which eliminates F_n as a triangular number. Then Table 2 establishes the result for $M = B$. The proofs for $M = C, D, E, F, G$, and H are given in Tables 3, 4, 5, 6, 7, and 8, respectively.

Table 1

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+20	X+24	X+39
0							31	31	9
40	2521	9	31	61	31	31		2521	9
80	31	9	2521	31	31	2521	2521	61	

Table 2

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+A/2	X+A-1
0							911	29
A	421	29	71	1427	71	911	911	71
2A	911	29	911	71	71	911	13	29
3A	83	29	13	71	281	83		29
4A	911	29	71	83	281	281	421	29
5A	911	71	71	71	911	13	911	29
6A	13	29	71	911	13	911	281	

Table 3

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+B/2	X+B-1
0							19	19
B	19	19	19	19	17	17	7	19
2B	19	19	17	17	19	19	19	167
3B		167	167	7	241	23	19	19
4B	19	19	19	19	17	17	167	19
5B	19	19	17	17	19	19	19	

Table 4

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+C/2	X+C-1
0							881	89
C	89	199	89	89	89	89	43	199
2C	43	199	89	43	881	307	199	199
3C	89	199	89	89	43	199	199	199
4C	331	199	881	661	199	199	199	199
5C	89	43	89	331	199	199		43
6C	881	199	307	199	199	307	89	199
7C	43	199	199	199	331	991	43	199
8C	199	199	199	199	89	89	89	199
9C	199	199	199	331	43	89	331	199
10C	199	89	307	89	89	89	89	

Table 5

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+D/2	X+D-1
0							151	3001
D	101	151	101	3001	101	47	3001	101
2D	3001	101	3001	3001	151	101	3041	101
3D	151	101	3001	3001	3001	47	101	151
4D	3001	3001	101	3001	3001	151	3001	1601
5D		1601	1601	1103	1103	47	151	3001
6D	101	151	101	3001	101	701	3001	101
7D	3001	101	3001	3001	151	101	1103	101
8D	151	101	3001	3001	3001	701	101	151
9D	3001	3001	101	3001	3001	151	3001	

Table 6

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+E/2	X+E-1
0							521	103
E	79	521	79	79	859	521	521	521
2E	1951	521	131	859	521	521	521	521
3E	131	859	2081	233	521	521	521	859
4E	79	233	859	521	521	521	521	103
5E	233	521	521	521	521	521	521	521
6E	521	79	521	521	521	521		79
7E	521	521	521	521	521	521	79	521
8E	521	103	521	521	521	1951	1951	233
9E	521	859	521	521	79	859	131	859
10E	521	521	521	521	2081	3329	79	521
11E	521	521	521	2081	79	79	3121	521
12E	521	103	79	859	131	859	521	

Table 7

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+F/2	X+F-1
0							239	919
F	3571	3571	3469	3571	3571	3469	3571	1597
2F	67	919	919	67	883	919	3469	3571
3F	1597	919	3571	919	1597	3571	1597	67
4F	1597	3469	919	1021	3469	3469	919	3571
5F	3571	3571	3469	3571	3571	1597	3571	67
6F	239	1597	919	67	67	1597	919	373
7F	919	3571	3571	1597	3469	3571	919	1597
8F	919	1597	3469	1597	919	239		1597
9F	1871	1597	919	3571	3571	919	3571	3571
10F	3571	373	373	3469	919	67	67	1597
11F	3469	67	1223	239	1597	3571	1597	3571
12F	1597	3571	3571	3469	1597	239	1597	3469
13F	919	67	3571	919	3571	919	3571	919
14F	3571	3571	1597	3571	3469	1223	239	919
15F	919	1597	1597	3469	919	3571	919	3571
16F	919	919	3571	1597	67	3469	919	

Table 8

X \ n	X	X+1	X+2	X+4	X+8	X+10	X+G/2	X+G-1
0							113	9349
G	113	113	37	9349	9349	229	9349	9349
2G	37	9349	37	9349	9349	37	229	797
3G	229	37	37	9349	37	9349	9349	37
4G	797	9349	9349	113	227	227	9349	9349
5G	191	9349	9349	9349	37	37	37	37
6G	229	229	9349	229	9349	37	2281	9349
7G	9349	37	229	9349	229	229	37	9349
8G	9349	761	9349	9349	113	37	9349	37
9G	9349	37	37	37	37	37		37
10G	113	37	9349	229	419	37	113	761
11G	9349	9349	9349	37	761	9349	37	37
12G	229	9349	191	9349	191	9349	229	229
13G	9349	37	683	227	229	9349	797	9349
14G	9349	9349	37	113	9349	229	191	9349
15G	37	37	9349	37	9349	9349	229	37
16G	2281	797	229	229	9349	37	9349	9349
17G	37	9349	37	229	113	9349	9349	113
18G	9349	9349	37	227	9349	9349	9349	

REFERENCES

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