GUESSING EXACT SOLUTIONS

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A recent problem [1] in this journal provides a nice illustration of a technique for guessing exact solutions of polynomial equations from approximate solutions. The technique depends on nothing more complicated than the familiar fact that if $ax^2 + bx + c = 0$ has roots s and t, then $s + t = -b/a$ and $st = c/a$.

Problem H-335 asked for exact solutions of the equation

$$x^5 - 5x^3 + 5x - 1 = 0. \tag{1}$$

One of the solutions is $x = 1$, and dividing (1) by $x - 1$ yields

$$x^4 + x^3 - 4x^2 - 4x + 1 = 0. \tag{2}$$

Using bracketing techniques and a calculator, it is relatively easy to see that (2) has rounded solutions: $r_1 = -1.8271$, $r_2 = -1.3383$, $r_3 = 0.2091$, $r_4 = 1.9563$.

Now we seek pairs of these solutions that have recognizable sums and products. Fibonacci fans are certainly familiar with the number $\alpha = (1 + \sqrt{5})/2 = 1.6180\ldots$. Upon noting that $r_2 + r_4 \approx 0.618 \approx \alpha^{-1}$ and $r_2 r_4 \approx -2.618 \approx -\alpha^2$, we suspect that r_2 and r_4 are solutions of

$$x^2 - \alpha^{-1}x - \alpha^2 = 0. \tag{3}$$

Long division, using familiar properties of powers of α, confirms that suspicion as fact, since

$$x^4 + x^3 - 4x^2 - 4x + 1 = (x^2 - \alpha^{-1}x - \alpha^2)(x^2 + \alpha x - \alpha^{-2}).$$

Then we can verify that r_2 and r_4 are indeed solutions of (3), namely,

$$x = \frac{\alpha^{-1} \pm \sqrt{\alpha^{-2} + 4\alpha^2}}{2} = \frac{\alpha - 1 \pm \sqrt{6 + 3\alpha}}{2} = \frac{-1 + \sqrt{5} \pm \sqrt{30 + 6\sqrt{5}}}{4}.$$

Also, r_1 and r_3 are solutions of $x^2 + \alpha x - \alpha^{-2} = 0$, namely,

$$x = \frac{-\alpha \pm \sqrt{\alpha^2 + 4\alpha^{-2}}}{2} = \frac{-\alpha \pm \sqrt{9 - 3\alpha}}{2} = \frac{-1 - \sqrt{5} \pm \sqrt{30 - 6\sqrt{5}}}{4}.$$

(Incidentally, the published solution was incorrect in that r_1 and r_3 were each off by 0.5, because of an incorrect sign in the numerator.)

REFERENCE

1. Paul Bruckman. Advanced Problem H-335. *The Fibonacci Quarterly* 20, no. 1 (1982):93.

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