DEFINITIONS

The Fibonacci numbers $F_n$ and Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1,$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$ 

PROBLEMS PROPOSED IN THIS ISSUE

B-544 Proposed by Herta T. Freitag, Roanoke, VA

Show that $p_{2n+1}^2 \equiv L_{2n+1}^2 \pmod{12}$ for all integers $n$.

B-545 Proposed by Herta T. Freitag, Roanoke, VA

Show that there exist integers $a$, $b$, and $c$ such that

$$F_{4n} \equiv an \pmod{5} \quad \text{and} \quad F_{4n+2} \equiv bn + c \pmod{5}$$

for all integers $n$.

B-546 Proposed by Stuart Anderson, East Texas State University, Commerce, TX and John Corvin, Amoco Research, Tulsa, OK

For positive integers $a$, let $S_a$ be the finite sequence $a_1, a_2, \ldots, a_n$ defined by

$$a_1 = a,$$

$$a_{i+1} = a_i / 2 \text{ if } a_i \text{ is even, } a_{i+1} = 1 + a_i \text{ if } a_i \text{ is odd},$$

the sequence terminates with the earliest term that equals 1.

For example, $S_5$ is the sequence 5, 6, 3, 4, 2, 1, of six terms. Let $K_n$ be the number of positive integers $a$ for which $S_a$ consists of $n$ terms. Does $K_n$ equal something familiar?
B-547 Proposed by Philip L. Mana, Albuquerque, NM

For positive integers \( p \) and \( n \) with \( p \) prime, prove that
\[
L_n^p \equiv L_nL_p \pmod{p}.
\]

B-548 Proposed by Valentina Bakinova, Rondout Valley, NY

Let \( D(n) \) be defined inductively for nonnegative integers \( n \) by \( D(0) = 0 \) and
\[
D(n) = 1 + D(n - [\sqrt{n}]^2),
\]
where \( [x] \) is the greatest integer in \( x \). Let \( n_k \) be the
smallest \( n \) with \( D(n) = k \). Then
\[
\begin{align*}
\eta_0 &= 0, \quad \eta_1 = 1, \\
\eta_2 &= 2, \quad \eta_3 = 3, \quad \text{and} \quad \eta_4 = 7.
\end{align*}
\]
Describe a recursive algorithm for obtaining \( \eta_k \) for \( k \geq 3 \).

B-549 Proposed by George N. Philippou, Nicosia, Cyprus

Let \( H_0, H_1, \ldots \) be defined by \( H_0 = q - p \), \( H_1 = p \), and \( H_{n+2} = H_{n+1} + H_n \) for
\( n = 0, 1, \ldots \). Prove that, for \( n \geq m \geq 0 \),
\[
H_{n+1}H_n - H_{n+2}H_{n+1} = (-1)^{n+1}[pH_{n-m+2} - qH_{n-m+1}].
\]

SOLUTIONS

Coded Multiplication Modulo 10 or 12

B-520 Proposed by Charles R. Wall, Trident Technical College, Charleston, SC

(a) Suppose that one has a table for multiplication \( \pmod{10} \) in which \( a \), \( b \), \ldots , \( j \) have been substituted for 0, 1, \ldots , 9 in some order. How many de-
codings of the substitution are possible?

(b) Answer the analogous question for a table of multiplication \( \pmod{12} \).

Solution by the proposer.

(a) There are two ways to decode the substitution. The letters represent-
ing 0 and 1 are easy to find, since \( x \cdot 0 = 0 \) and \( x \cdot 1 = x \) for all \( x \); then 9 is
easily found as the unique solution to \( x^2 - 1 \) with \( x \neq 1 \). The letter repre-
senting 5 is identifiable, and the letters are easily sorted as odd or even,
because \( 5 \cdot x = 5 \) if \( x \) is odd and \( 5 \cdot x = 0 \) if \( x \) is even. Then 6 is identified
from \( 6 \cdot x = x \) if \( x \) is even, and 4 is identified from \( x^2 = 6 \) with \( x \neq 6 \). Still
unidentified are 2, 3, 7, and 8, but \( 2^2 = 8^2 = 4 \) and \( 3^2 = 7^2 = 9 \), so there are
two choices for 3. Once 3 is chosen, 7 is forced, and so are 2 and 8, since
\( 3 \cdot 4 = 2 \) and \( 3 \cdot 6 = 8 \).

(b) The substitution is unique. As in (a), 0 and 1 are easily identified.
Then \( 6 \) is easily found, and the letters can be classified as odd or even,
because \( 6 \cdot x = 6 \) if \( x \) is odd and \( 6 \cdot x = 0 \) if \( x \) is even. Now, 4 is the only non-
zero even solution of \( x^2 = x \). If \( x \) and \( y \) are both even, then \( x \cdot y \) is 0, 4, or
8, and since 0 and 4 are already known, 8 is easily identified, leaving only 2
and 10 unknown among the even numbers. But \( 8 \cdot 2 = 4 \) and \( 8 \cdot 10 = 8 \), so 2 and 10
can be determined. Among the odd numbers, 9 is the only solution to \( x^2 = x \)
with \( x \neq 1 \), so 9 is easily identified. If \( x \) is odd, then \( 9 \cdot x \) is either 3 or 9, so
3 is determined. Then 7 is identified using the fact that \( 7 \cdot x = x \) if \( x \) is
even. To identify 5 and 11, we use the fact that \( 3 \cdot 5 = 3 \), while \( 3 \cdot 11 = 9 \).
ELEMENTARY PROBLEMS AND SOLUTIONS

Also solved by Paul S. Bruckman and by L. Kuipers & P. A. J. Scheelbeek.

Unique Decoding

B-521 Proposed by Charles R. Wall, Trident Technical College, Charleston, SC

See the previous problem. Find all moduli \( m > 1 \) for which the multiplication (mod \( m \)) table can be decoded in only one way.

Solution by the proposer.

Suppose the multiplication (mod \( m \)) table can be decoded uniquely. Then it is easy to see that if \( K|m \), the multiplication (mod \( K \)) table can also be decoded in only one way.

If \( p \geq 5 \) is prime, there are at least two distinct primitive roots (mod \( p \)), say \( g \) and \( h \); replacing \( g^n \) by \( h^n \) for each \( n \) yields an equivalent substitution, so the multiplication (mod \( p \)) table cannot be decoded uniquely, and hence \( p \nmid m \).

The multiplication (mod 9) table cannot be decoded uniquely, because 3 and 6 may be interchanged, and in the multiplication (mod 8) table, 2 and 6 may be switched.

Therefore, \( m = 2^a 3^b \) with \( a \leq 2 \) and \( b \leq 1 \). Since the multiplication (mod 12) table can be decoded in only one way, \( m = 2, 3, 4, 6 \), or 12.

Also solved by Paul S. Bruckman and by L. Kuipers & P. A. J. Scheelbeek.

Alternating Even and Odd

B-522 Proposed by Joan Tomescu, University of Bucharest, Romania

Find the number \( A(n) \) of sequences \( (a_1, a_2, \ldots, a_k) \) of integers \( a_i \) satisfying \( 1 \leq a_i < a_{i+1} \leq n \) and \( a_{i+1} - a_i \equiv 1 \) (mod 2) for \( i = 1, 2, \ldots, k - 1 \). [Here \( k \) is variable but, of course, \( 1 \leq k \leq n \). For example, the three allowable sequences for \( n = 2 \) are \((1, 2)\), \((2, 1)\), and \((1, 2)\).]

Solution by J. Suck, Essen, Germany

\[
A(n) = F_{n+3} - 2.
\]

Proof by Double Induction

Let \( B(n) \) be the number of sequences of the said type with \( a_1 = n \). I claim that \( B(n) = F_{n+1} \). This is so for \( n = 1, 2 \). Suppose it is true for \( n = 1, \ldots, n - 1 \geq 1 \). The sequences with \( a_k = n \), except \((n)\), consist of those with \( a_{k-1} = n - 1 \) or \( n - 3 \) or \( n - 5 \) \ldots . Thus

\[
B(n) = 1 + F_n + F_{n-2} + \cdots + \begin{cases} 
F_2 & \text{for } n \text{ even} \\
F_3 & \text{for } n \text{ odd}
\end{cases}
\]

= \( F_{n+1} \) in any case by Hoggatt's \( I_5 \) or \( I_6 \).

Now, \( A(1) = 1 = F_4 - 2 \), and, clearly,

\[
A(n) = A(n - 1) + B(n) = F_{n+2} - 2 + F_{n+1} = F_{n+3} - 2 \text{ for } n > 1.
\]

Reversing Coefficients of a Polynomial

Let \( p, a_0, a_1, \ldots, a_n \) be integers with \( p \) a positive prime such that
\[
gcd(a_0, p) = 1 = gcd(a_n, p).
\]
Prove that in \( \{0, 1, \ldots, p-1\} \) there are as many solutions of the congruence
\[
a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \equiv 0 \pmod{p}
\]
as there are of the congruence
\[
a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \equiv 0 \pmod{p}.
\]

Solution by Sahib Singh, Clarion University of Pennsylvania, Clarion, PA

Since \( gcd(a_0, p) = gcd(a_n, p) = 1 \), it follows that both polynomials associated with the given congruences are of \( n \)th degree and that zero is not a solution of any one of these congruences. If \( \alpha \) is a solution of the first congruence, then \( \alpha^{-1} \) is a solution of the second congruence where \( \alpha^{-1} \) denotes the unique multiplicative inverse of \( \alpha \) in \( \mathbb{Z}_p \).

Thus, we conclude that if \( \alpha_1, \alpha_2, \ldots, \alpha_s \) are the solutions of the first congruence in \( \mathbb{Z}_p \), then \( \alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_s^{-1} \) are precisely the solutions of the second congruence in \( \mathbb{Z}_p \).

Also solved by Paul S. Bruckman, L. A. G. Dresel, L. Kuipers, J. M. Metzger, Bob Priellipp, and the proposer.

Disguised Fibonacci Squares

Let \( S_n = F_{2n-1}^2 + F_n F_{n-1} (F_{2n-1} + F_n^2) + 3F_n F_{n+1} (F_{2n-1} + F_n F_{n-1}) \).

Show that \( S_n \) is the square of a Fibonacci number.

Solution by Paul S. Bruckman, Fair Oaks, CA

Let \( a = F_n, b = F_{n+1} \). Note that \( F_{2n-1} = a^2 + b^2, F_{n+1} = a + b \). Then
\[
S_n = (a^2 + b^2)^2 + ab(a^2 + b^2 + a^2) + 3a(a + b)(a^2 + b^2 + ab)
\]
\[
= a^4 + 2a^3b^2 + b^4 + 2a^3b + ab^3 + 3a^4 + 6a^3b + 6a^2b^2 + 3ab^3
\]
\[
= 4a^4 + 8a^3b + 8a^2b^2 + 4ab^3 + b^4
\]
\[
= (2a^2 + 2ab + b^2)^2.
\]

Now \( 2a^2 + 2ab + b^2 = a^2 + (a + b)^2 = F_n^2 + F_{n+1}^2 = F_{2n+1} \). Hence, \( S_n = F_{2n+1}^2 \).

Also solved by L. A. G. Dresel, L. Kuipers, Imre Merenyi, J. M. Metzger, Bob Priellipp, Sahib Singh, J. Suck, and the proposer.
Diophantine Equation

B-252 Proposed by Walter Blumberg, Coral Springs, FL

Let $x, y,$ and $z$ be positive integers such that $2^z - 1 = y^z$ and $x > 1$. Prove that $z = 1$.

Solution by Leonard A. G. Dresel, University of Reading, England

Since $x > 1$, we have $y^z = 2^z - 1 \equiv -1 \pmod{4}$. Hence, $y \equiv -1 \pmod{4}$ and $z$ is odd, so that we have the identity

$$y^z + 1 = (y + 1)(y^{z-1} - y^{z-2} + \cdots - y + 1).$$

Hence, $y + 1$ divides $y^z + 1 = 2^z$, so that $y + 1 = 2^u$, $u \leq x$, and

$$2^{z-u} = y^{z-1} - y^{z-2} + \cdots - y + 1 \equiv 1 + 1 + \cdots + 1 + 1 \pmod{4}$$

$$\equiv z \pmod{4} \quad \text{(since there are $z$ terms)}$$

$$\equiv 1 \pmod{2}, \text{ since } z \text{ is odd.}$$

Therefore, we must have $x - u = 0$, and $y^z = y$, and since $y^z > 1$ it follows that $z = 1$.

Note by Paul S. Bruckman

This is apparently a well-known result, indicated by S. Ligh and L. Neal in "A Note on Mersenne Numbers," Math. Magazine 47, no. 4 (1974):231-33. The result indicated in that reference is that a Mersenne number cannot be a power (greater than one) of an integer.


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 ELEMENTARY PROBLEMS AND SOLUTIONS