EUCLID'S ALGORITHM AND THE FIBONACCI NUMBERS

P. W. EPASINGHE
University of Colombo, Colombo 3, Sri Lanka

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The number of steps in Euclid's algorithm for the natural number pair (a, b) with $a > b$ is discussed. If the number of steps is k, then the least possible value for a is F_{k+2}. If the number of steps exceeds k, then $a \geqslant F_{k+3}$. If the number of steps is k and $a = F_{k+2}$, then $b = F_{k+1}$. If $b = F_{k+1}$ and the number of steps is k, then $a = F_k + nF_{k+1}$ where n is any natural number. (F_k is the k^{th} Fibonacci number.)

Given two natural numbers a, b, Euclid's algorithm produces the greatest common divisor of a and b. The Fibonacci numbers are defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$ where n is a natural number, with $F_1 = F_2 = 1$. Various interesting properties of these numbers can be found in the literature. In the following, we shall demonstrate an extremal property of the Fibonacci numbers in relation to Euclid's algorithm.

If the n^{th} quotient and n^{th} remainder in Euclid's algorithm are q_n and r_n, respectively, and the algorithm consists of at least k steps, then the sequence of steps up to and including the k^{th} step can be written algebraically as follows:

$$\left.\begin{array}{l} r_{n-2} = q_n r_{n-1} + r_n, \quad n = 1, 2, 3, \ldots, k; \\ \text{where } r_{-1} = a, \quad r_0 = b. \end{array}\right\} \tag{1}$$

Further, all the quantities r_{n-2}, r_{n-1}, r_n, q_n are natural numbers except r_k, which may also be zero.

Therefore, given any two natural numbers a, b with $a > b$, there is a unique natural number $e(a, b)$ associated with them where $e(a, b)$ is the number of operations in Euclid's algorithm for the greatest common divisor of a and b. We have, for example, $e(a, 1) = 1$ for all natural numbers a (> 1).

Given any natural number k, it is possible to determine a pair of natural numbers a, b with $a > b$ such that $e(a, b) = k$. This is not obvious for all k, but will be seen in a little while to be true. As special cases—$e(2, 1) = 1$, $e(3, 2) = 2$, $e(5, 3) = 3$, and $e(8, 5) = 4$—and it can be shown that all these number pairs are consecutive Fibonacci numbers. As a generalization, it follows that

$$e(F_{k+2}, F_{k+1}) = k. \tag{2}$$

Given k, the number of pairs (a, b) such that $e(a, b) = k$ is nonfinite because, for all natural numbers n, $e(a + nb, b) = e(a, b)$. As a special consequence, we also have

$$e(F_{k+3}, F_{k+1}) = k. \tag{3}$$

It now follows that, given a natural number k,

$$\{a \mid e(a, b) = k \text{ for some natural number } b < a\}$$

is not bounded above, but being a subset of the set of natural numbers should have a least element. It is convenient to denote this least element by $e(k + 2)$

with $e(1) = e(2) = 1$. We will also call $e(k)$ the Euclid number of k. The main result that justifies the title of this note is:

"The Euclid number of the natural number k is the k^{th} Fibonacci number."

Before proving this result, we need an equation that we shall be using over and over again. We multiply the equation in (1) corresponding to each value of n by F_n and sum over all the values of n. This yields

$$\sum_{n=1}^{k} F_n r_{n-2} = \sum_{n=1}^{k} F_n r_{n-1} q_n + \sum_{n=1}^{k} F_n r_n.$$

$$\therefore\ F_1 a + F_2 b + \sum_{n=1}^{k-2} F_{n+2} r_n = bq_1 + \sum_{n=1}^{k-2} F_{n+1} r_n q_{n+1} + F_k r_{k-1} q_k$$

$$+ \sum_{n=1}^{k-2} F_n r_n + F_{k-1} r_{k-1} + F_k r_k \quad \text{if } k \geqslant 3.$$

That is,

$$a = b(q_1 - 1) + \sum_{n=1}^{k-2} F_{n+1} r_n (q_{n+1} - 1) + F_k r_{k-1} q_k + F_{k-1} r_{k-1} + F_k r_k,$$

where we have used the fact that $F_{n+2} = F_{n+1} + F_n$ when $n = 1, 2, \ldots, k - 2$.

$$\therefore\ a - F_{k+1} = b(q_1 - 1) + \sum_{n=1}^{k-2} F_{n+1} r_n (q_{n+1} - 1)$$

$$+ F_k r_{k-1} q_k + F_{k-1}(r_{k-1} - 1) + F_k(r_k - 1);$$

$$\therefore\ a - F_{k+1} = b(q_1 - 1) + \sum_{n=1}^{k-1} F_{n+1} r_n (q_{n+1} - 1)$$

$$+ F_{k+1}(r_{k-1} - 1) + F_k r_k. \tag{4}$$

Equation (4) has been obtained only when $k \geqslant 3$. However, it is easily verified to be true even when $k = 2$.

Property 1

If the number of steps in Euclid's algorithm for the pair of natural numbers a, b, where $a > b$, is exactly k, then

$$a \geqslant F_{k+2}.$$

The case when $k = 1$ is trivial. When $k \geqslant 2$, we have $r_k = 0$ and $q_k \geqslant 2$. Also, $q_n \geqslant 1$, $n = 1, 2, \ldots, k - 1$, and $r_n \geqslant 1$, $n = 1, 2, \ldots, k - 1$. Hence, by equation (4),

$$a - F_{k+1} \geqslant F_k$$

$$\therefore\quad a \geqslant F_{k+2}.$$

Thus, the least value of a is $F_{k+2} = e(k + 2)$. This proves the main result as stated earlier.

Property 2

If the number of steps in Euclid's algorithm for the pair of natural numbers a, b, where $a > b$, is greater than k, then $a \geqslant F_{k+3}$.

Here again, the case when $k = 1$ is trivial. When $k \geqslant 2$, we have $r_k \geqslant 1$ and $r_{k-1} \geqslant 2$. Also, $q_n \geqslant 1$, $n = 1, 2, \ldots, k$. Equation (4) now gives

$$a - F_{k+1} \geqslant F_{k+1} + F_k = F_{k+2}$$

$$\therefore \quad a \geqslant F_{k+1} + F_{k+2} = F_{k+3}.$$

Property 3

If the number of steps in Euclid's algorithm for the pair of natural numbers F_{k+2}, b, where $F_{k+2} > b$, is exactly k, then

$$b = F_{k+1}.$$

Here again, the case $k = 1$ is trivial. When $k \geqslant 2$, $a = F_{k+2}$, $r_k = 0$, and $q_k \geqslant 2$, whereas $q_n \geqslant 1$ and $r_n \geqslant 1$ when $n = 1, 2, \ldots, k - 1$. Equation (4) now gives

$$0 = b(q_1 - 1) + \sum_{n=1}^{k-2} F_{n+1} r_n (q_{n+1} - 1) + F_{k+1}(r_{k-1} - 1) + F_k[r_{k-1}(q_k - 1) - 1] \quad \text{if } k \geqslant 2,$$

with obvious modifications if $k = 1$. Since this is the sum of a number of terms, each of which is nonnegative, each term should be zero.

$$\therefore q_n = 1,\ n = 1, 2, \ldots, k - 1; \quad r_{k-1} = 1 \quad \text{and} \quad q_k = 2.$$

Equation set (1) now reduces to

$$\left.\begin{aligned} r_{n-2} &= r_{n-1} + r_n,\ n = 1, 2, \ldots, k - 1, \\ r_{k-2} &= 2, \\ F_{k+2} &= r_{-1}. \end{aligned}\right\} \tag{5}$$

This set of equations has a unique solution with

$$r_n = F_{k+1-n},\ n = -1, 0, 1, \ldots, k - 2.$$

In particular, $r_0 = F_{k+1}$.

Property 4

If the number of steps in Euclid's algorithm for the pair a, F_{k+1}, where $a > F_{k+1}$, is k, then $a = F_k + nF_{k+1}$, where n is any natural number.

Here, too, the case when $k = 1$ is trivial. When $k \geqslant 2$, we can use Eq. set (1) directly. Leaving the equation corresponding to $n = 1$ out for the moment, the other $k - 1$ equations would correspond to a $(k - 1)$-step Euclid algorithm for the number pair F_{k+1}, r_1, where $r_1 < F_{k+1}$.
By an application of Property 3, $r_1 = F_k$.

$$\therefore a = bq_1 + r_1 = F_k + q_1 F_{k+1}, \text{ where } q_1 \text{ is any natural number.}$$

This proves the result.

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