A COROLLARY TO ITERATED EXPONENTIATION

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In connection with three previous papers on the convergence of iterated exponentiation by Creutz and Sternheimer [1], [2], [3], and with some earlier work [4], [5], it occurred to me that the problem of the proof of Fermat's Last Theorem might be intimately connected with the properties of the function $F(x, y) \equiv x^y - y^x$, and in particular with the condition that

$$F(x, y) = 0, \tag{1}$$

when x and y are restricted to be positive integers [6]. It can be shown that aside from the trivial solution $x = y$, (1) is satisfied only for $x = 2$, $y = 4$, in which case

$$F(x, y) = 2^4 - 4^2 = 0. \tag{2}$$

In order to prove this property of $F(x, y)$, we consider Figure 1 of [1]. This figure gives the function $f(x)$ defined by the condition

$$x^f = f. \tag{3}$$

In Figure 1 of [1], we consider the continuation of the dashed part of the curve to the right of $f(x) = e$ up to the region of $f(x) = 4$. It is easily seen that the corresponding x is $\sqrt{2}$, since $(\sqrt{2})^4 = 2^2 = 4$ satisfies (3).

We also have $f(x) = 2$ for $x = \sqrt{2}$, as shown by the left-hand part of Figure 1. If we denote the two values of $f(\sqrt{2})$ by f_1 and f_2, we have

$$x^{f_1} = f \ , \ x^{f_2} = f \ , \tag{4}$$

where $x = \sqrt{2}$. We can rewrite (4) as follows:

$$f_1^{1/f_1} = f_2^{1/f_2} = x = \sqrt{2}. \tag{5}$$

From (5), we obtain (by raising to the power $f_1 f_2$):

$$f_1^{f_2} = f_2^{f_1}, \tag{6}$$

i.e., $2^4 = 4^2$.

Thus the two values of $f(x)$ for a given x, namely f_1 and f_2, are the solutions of the equation $f_1^{f_2} = f_2^{f_1}$ (6). We can now set $f_1 = x$, $f_2 = y$ in the notation of (1) (where x is not to be confused with the auxiliary x of Figure 1 of [1]). Now, from Figure 1, it is obvious that one of the f's, say f_1, must be less than e, while the other f, say f_2, must be larger than e. It is also clear that, since the only integer smaller than e and larger than 1 is 2, the equation $f_1^{f_2} = f_2^{f_1}$ can be satisfied only for $f_1 = 2$, $f_2 = 4$, if f_1 and f_2 are restricted to be integers.

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Incidentally, Figure 2 of [1] shows that, when the ordinate x is less than 1, there is no second branch of the curve of x vs. f, and therefore, for $f_1 < 1$, there is no f_2 such that $f_1^{1/f_1} = f_2^{1/f_2}$.

The fact that $x = 2$, $y = 4$ is the only integer solution of $F(x, y) = 0$ can also be seen by inspection, i.e., by calculating

$$F(2, 3) = -1, \quad F(2, 4) = 0, \quad F(2, 5) = 7, \quad F(2, 6) = 28, \quad F(3, 4) = 17,$$

etc. Also, for arbitrary x and y such that the difference $y - x \equiv \Delta x$ is small, it can be shown by differentiation of x^y with respect to both x and y that

$$F(x, y) = \bar{x}^{\bar{x}}(\ell n\bar{x} - 1)(y - x), \tag{7}$$

where $\bar{x} \equiv (x + y)/2$. In order to prove (7), we note that

$$F(x, y) = x^{x+\Delta x} - (x + \Delta x)^x. \tag{8}$$

Now, if Δx is small, we can expand both terms in the right-hand side of (8) as follows, to first order in Δx:

$$x^{x+\Delta x} = x^x + x^x \ln x\Delta x, \tag{9}$$

where we have used $\partial x^y/\partial y = x^y \ln x$. Moreover,

$$(x + \Delta x)^x = x^x + x^x\Delta x, \tag{10}$$

where we have used

$$\partial x^y/\partial x = yx^{y-1} = \frac{y}{x}x^y \approx x^y. \tag{11}$$

Upon subtracting (10) from (9), one finds:

$$F(x, y) = x^x(\ln x - 1)\Delta x = x^x(\ln x - 1)(y - x). \tag{12}$$

Because of the rapid increase of x^x with increasing x, one will obtain a more accurate result by evaluating the derivatives $\partial x^y/\partial y$ and $\partial x^y/\partial x$ at the midpoint of the interval (x, y), i.e., at the point $\bar{x} = (x + y)/2$. Upon making this substitution in (12), one obtains (7).

Equation (7) shows that for $y - x$ small, x^y is *larger* than y^x for positive Δx if $\bar{x} > e$ and is *smaller* than y^x for positive Δx if $\bar{x} < e$. As an example, $1.6^{1.7} = 2.2233$ is smaller than $1.7^{1.6} = 2.3373$ because $1.6, 1.7 < e$. The difference $F(1.6, 1.7) = -0.1140$ is very well reproduced by (7), which gives, with $\bar{x} = 1.65$:

$$F(1.6, 1.7) = 1.65^{1.65}(\ln 1.65 - 1)(0.1) = -0.1140. \tag{13}$$

As a second example, $2.9^{3.0} = 24.389$ is larger than $3.0^{2.9} = 24.191$ because $2.9, 3.0 > e$. We find $F(2.9, 3.0) = 24.389 - 24.191 = +0.198$, and this difference is very well reproduced by (7), which gives, with $\bar{x} = 2.95$:

$$F(2.9, 3.0) = 2.95^{2.95}(\ln 2.95 - 1)(0.1) = +0.199. \tag{14}$$

Equation (7) again points out the crucial role of the constant e for the sign of $F(x, y)$, since $\ln \bar{x} - 1 = \ln(\bar{x}/e)$. The same equation also shows that for x and y close to e and $x < e$, $y > e$, we must have

$$\bar{x} = (1/2)(x + y) = e \quad \text{for } F(x, y) = 0.$$

Obviously, (7) does not hold when the difference $y - x$ is large, and the previous result $x = 2$, $y = 4$ with $x < e$, $y > e$ can be regarded as an extreme example of (7) when higher derivatives of x^y, i.e., terms in $(\Delta x)^2$, $(\Delta x)^3$, etc., are included.

It is of interest to speculate that $x^n + y^n = z^n$ is solvable only for $n = 1$ and $n = 2$ (with x, y, z = positive integers) because $n = 1$ and $n = 2$ are the only positive integers smaller than e. Here I wish to mention that the Fermat equation $x^n + y^n = z^n$ has solutions both for $n = 1$ and $n = 2$. The case $n = 2$ has been discussed frequently; however, the case $n = 1$ also merits some attention. Thus, if we assume (by definition) that $x \geqslant y$, then $x + y = z$ has $z/2$ distinct solutions when z = even, and it has $(z - 1)/2$ distinct solutions when z = odd. As an example for $z = 11$, we have five distinct solutions:

$$x + y = 6 + 5,\ 7 + 4,\ 8 + 3,\ 9 + 2,\text{ and } 10 + 1.$$

In this connection, I wish to point out that in complete analogy to the exponent n which appears in the Fermat equation, the equation $F(x, y) = 0$, in addition to $F(2, 4) = 0$, also has a valid solution for $x = 1$, namely $F(1, y) = 0$ in the limit in which y approaches infinity. This additional solution will be discussed in detail in a forthcoming paper.

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6. I have also considered the properties of the function $G(x, y) \equiv x^y + y^x$. It is of interest that $G(2, 6) = 100$.

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