ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-389 *Proposed by Andreas N. Philippou, University of Patras, Patras, Greece*

Show that

$$F_{n+2}^{(n-i)} = 2^n - 2^i(1 + i/2) \qquad (n \geqslant 2i + 1)$$

for each nonnegative integer i, where $F_{n+2}^{(n-i)}$ is the $n + 2$ Fibonacci number of order $n - i$ [1] and $F_3^{(1)} = 1$.

Reference

1. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order k." *The Fibonacci Quarterly 20*, no. 1 (1982):28-32.

H-390 *Proposed by M. Wachtel, Zurich, Switzerland*

For every m,

$2F_{2-m}F_{5+m} + (-1)^m(F_mF_{m+1} + F_{m+2}^2)$ has the unique value 11.

Find a general formula for analogous constant values, which should represent the terms of an infinite sequence.

Prove that no divisor of any of these terms is congruent to 3 or 7 modulo 10.

H-391 *Proposed by Lawrence Somer, Washington, D.C.*

For every n, show that no integral divisor of L_{2n} is congruent to 11, 13, 17, or 19 modulo 20. (This problem was suggested by Problem H-364 on p. 313 of the November 1983 issue of *The Fibonacci Quarterly*.)

SOLUTIONS

Any More?

H-363 *Proposed by Andreas N. Philippou, University of Patras, Patras, Greece (Vol. 21, no. 4, November 1983)*

For each fixed integer $k \geqslant 2$, let $\left\{f_n^{(k)}\right\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k, i.e., $f_0^{(k)} = 0$, $f_1^{(k)} = 1$, and

$$f_n^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_0^{(k)}, & \text{if } 2 \leqslant n \leqslant k, \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)}, & \text{if } n \geqslant k+1. \end{cases}$$

Evaluate the series

$$\sum_{n=0}^{\infty} \frac{1}{f_{m^n}^{(k)}} \qquad (k \geqslant 2,\ m \geqslant 2).$$

Remark: The Fibonacci sequence of order k appears in the work of Philippou and Muwafi [*The Fibonacci Quarterly 20* (1982):28-32.]

Comment by Paul S. Bruckman, Carmichael, CA

Letting

$$S(k, m) = \sum_{n=0}^{\infty} (f_{m^n}^{(k)})^{-1},$$

to the best of my knowledge, the only known result (fairly well-known in fact), is

$$S(2, 2) = \sum_{n=0}^{\infty} 1/F_{2^n} = \tfrac{1}{2}(7 - \sqrt{5}) \doteq 2.381966.$$

I would be very surprised—indeed, amazed!—to learn of any other closed form expressions for $S(k, m)$.

Only Two!

H-364 *Proposed by M. Wachtel, Zurich, Switzerland (Vol. 21, no. 4, November 1983)*

For every n, show that no integral divisor of L_{2n+1} is congruent to 3 or 7 modulo 10.

Solution by Paul S. Bruckman, Carmichael, CA

Given any prime p with $p \equiv \pm 3 \pmod{10}$, then $(5/p) = (p/5) = -1$. It is sufficient to prove that p does not divide L_{2n+1} for all n, since any divisor of L_{2n+1} congruent to 3 or 7 (mod 10) must be divisible by such a prime. By the calculus of "complex residues" (see [1]),

$$\alpha^p \equiv \beta,\ \beta^p \equiv \alpha \pmod{p}. \tag{1}$$

This, in turn, inplies $\alpha^{p+1} \equiv \beta^{p+1} \equiv -1 \pmod{p}$; hence,

$L_{p+1} \equiv -2 \pmod{p}, \quad F_{p+1} \equiv 0 \pmod{p}.$

In the sequel all congruences will be understood to be modulo p, and the notation "(mod p)" will be omitted wherever no confusion is likely to arise. We will let $e = e(p)$ denote the "entry point" (if any) of p in the Lucas sequence, i.e., e is the smallest positive integer k (if any) such that $L_k \equiv 0 \pmod{p}$. We consider two distinct cases:

(A) $p \equiv 3$ or $7 \pmod{20}$. Let $s = \frac{1}{4}(p + 1)$, an integer. Then

$$(-1)^{\frac{1}{2}(p+1)} = (-1)^{2s} = 1.$$

Note that $L_{p+1} = L_{4s} = L_{2s}^2 - 2 \equiv -2$. Hence,

$$L_{2s} \equiv 0. \tag{3}$$

Thus, e exists and we must have

$$e \mid 2s. \tag{4}$$

We suppose e is odd. Then, since $L_e \equiv 0$, we must have $L_{me} \equiv 0$ for all odd m, because $L_e \mid L_{me}$ in that case. On the other hand,

$$L_{2e} = L_e^2 + 2 \equiv 2, \quad L_{4e} = L_{2e}^2 - 2 \equiv 2, \quad L_{6e} = L_{3e}^2 + 2 \equiv 2, \text{ etc.},$$

i.e., $L_{me} \equiv 2$ for all even m. Since $2s$ is an even multiple of e, it follows that $L_{2s} \equiv 2$, which is a contradiction of (3); thus, e is even. Now, given any positive k with $L_k \equiv 0$, we have $e \mid k$. Since e is even, so is k. Therefore, the congruence $L_{2n+1} \equiv 0$ is impossible in this case.

(B) $p \equiv 13$ or $17 \pmod{20}$. We will show that $L_k \not\equiv 0$ for all k, in this case, i.e., e does not exist. Let e' denote the entry point of p in the Fibonacci sequence, i.e., e' is the smallest positive integer k with $F_k \equiv 0 \pmod{p}$. It is known (see [2]) that e' always exists and that, if e exists, then $e' = 2e$. We suppose e exists; hence, e' is even.

Let $t = \frac{1}{2}(p + 1)$, an odd number. Then $L_t^2 + 2 = L_{2t} = L_{p+1} \equiv -2$, which implies $L_t \not\equiv 0$. Also, since $F_{p+1} = F_{2t} = F_t L_t \equiv 0$, we have $F_t \equiv 0$. Therefore, $e' \mid t$. However, because t is odd, it cannot be divisible by an even integer. This contradiction establishes that e does not exist. Hence, $L_k \not\equiv 0$ for all k, in this case; a fortiori, the congruence $L_{2n+1} \equiv 0$ is impossible.

Combining the results of (A) and (B), we reach the desired conclusion.

REFERENCES

1. P. S. Bruckman. "Some Divisibility Properties of Generalized Fibonacci Sequences." *The Fibonacci Quarterly 17*, no. 1 (1979):42-49.

2. Brother A. Brousseau (compiler). *Fibonacci and Related Number Theoretic Tables*, p. 25. Santa Clara, Calif: The Fibonacci Association, 1972.

Also solved by L. Somer and the proposer.

Poly Nomial

H-366 *Proposed by Stanley Rabinowitz, Digital Equipment Corp. Merrimack, NH (Vol. 22, no. 1, February 1984)*

The *Fibonacci Polynomials* are defined by the recursion

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$$

with the initial conditions $f_1(x) = 1$ and $f_2(x) = x$. Prove that the discriminant of $f_n(x)$ is

$$(-1)^{(n-1)(n-2)/2}2^{n-1}n^{n-3} \text{ for } n > 1.$$

Remark: The idea of investigating discriminants fo interesting polynomials was suggested by [1]. The definition of the discriminant of a polynomial can be found in [2]. Fibonacci polynomials are well known (see, e.g., [3] and [4]). I ran a computer program to find the discriminant of $f_n(x)$ as n varied from 2 to 11, and by analyzing the results, reached the conjecture given above in the proposed problem. The discriminant was calculated by finding the resultant of $f_n(x)$ and $f_n'(x)$ using a computer algebra system similar to the MACSYMA program as described in [5]. Much useful material can be found in [6] where the problem of finding the discriminant of the Hermite, Laguerre, and Chebyshev polynomials is discussed. The discriminant of the Fibonacci polynomials should be provable using similar techniques; however, I was not able to do so.

REFERENCES

1. Phyllis Lefton. "A Trinomial Discriminant Formula." *The Fibonacci Quarterly 20*, no. 4 (1982):363-365.
2. Van der Warden. *Modern Algebra*, Vol. I, p. 82. New York: Ungar, 1940.
3. M. N. S. Swamy. Problem B-84. *The Fibonacci Quarterly 4* (1966):90.
4. Stanley Rabinowitz. Problem H-129. *The Fibonacci Quarterly 6* (1968):51.
5. W. A. Martin & R. J. Fateman. "The MACSYMA System." *Proceedings of the 2nd Symposium on Symbolic and Algebraic Manipulation*, pp. 59-75. Association for computing Machinery, 1971.
6. D. K. Faddeev & I. S. Sominskii. *Problems in Higher Algebra*. Trans. by J. L. Brenner. San Francisco: Freeman and Company. Problems 833-851.

Solution by Paul S. Bruckman, Carmichael, CA

The Fibonacci polynomials are given by the explicit expression

$$f_n(x) = \frac{u^n - v^n}{u - v}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where

$$u = u(x) = \tfrac{1}{2}(x + \sqrt{x^2 + 4}), \quad v = v(x) = \tfrac{1}{2}(x - \sqrt{x^2 + 4}). \tag{2}$$

From the defining recursion and the initial values, it is easy to see that f_n is a monic polynomial of degree $n - 1$.

We also define the *Lucas polynomials* $g_n(x)$ as follows:

$$g_n(x) = u^n + v^n, \quad n = 0, 1, 2, \ldots . \tag{3}$$

Let

$$D_n = \text{disc}(f_n), \quad n = 2, 3, \ldots . \tag{4}$$

If the zeros of f_n are $x_1, x_2, \ldots, x_{n-1}$, an explicit expression for D_n is given by

$$D_n = \prod_{1 \leqslant r < s \leqslant n-1} (x_r - x_s)^2, \quad n \geqslant 3; \quad \text{also, } D_2 = 1. \tag{5}$$

We also know from higher algebra that, if the x_k's are distinct,

$$|D_n| = \left|\prod_{k=1}^{n-1} f'(x_k)\right|. \tag{6}$$

We will use (5) only to determine the sign of D_n, and (6) to determine its absolute value, using the relation

$$D_n = |D_n| \cdot \operatorname{sgn}(D_n).$$

The x_k are determined by setting the expression in (1) equal to zero. Then

$$(u/v)^n = 1 \Rightarrow u/v = \exp(2ki\pi/n);$$

since $uv = -1$, we have

$$-u^2 = \exp(2ki\pi/n) \Rightarrow u = \pm i \exp(ki\pi/n).$$

Changing the sign in the last expression above is equivalent to replacing k by $(n - k)$, showing that we need to consider only the positive sign. Thus, we may take $u = i \exp(ki\pi/n)$; hence, $v = i \exp(-ki\pi/n)$. Since f_n is of degree $n - 1$, we may take k to vary from 1 through $n - 1$; thus,

$$x_k = u + v = 2i \cos(k\pi/n), \quad k = 1, 2, \ldots, n - 1.$$

Note that the x_k are distinct, which allows the use of (6). Finally, since f_n is monic and a polynomial, we obtain the factorization

$$f_n(x) = \prod_{k=1}^{n-1} (x - 2i \cos(k\pi/n)), \quad n = 2, 3, \ldots . \tag{7}$$

To evaluate the expression in (6), we differentiate (1), noting first that

$$u'(x) = \tfrac{1}{2}(1 + x/\sqrt{x^2 + 4}), \quad v'(x) = \tfrac{1}{2}(1 - x/\sqrt{x^2 + 4})$$

or

$$u'(x) = \frac{u}{u - v}, \quad v'(x) = \frac{-v}{u - v}. \tag{8}$$

Then,

$$f_n'(x) = \frac{(u - v)\left\{\dfrac{nu^{n-1} \cdot u + nv^{n-1} \cdot v}{u - v}\right\} - (u^n - v^n)\left\{\dfrac{u + v}{u - v}\right\}}{(u - v)^2}$$

$$= \frac{n(u^n + v^n) - x\left\{\dfrac{u^n - v^n}{u - v}\right\}}{(u - v)^2},$$

or

$$f_n'(x) = \frac{ng_n(x) - xf_n(x)}{x^2 + 4}. \tag{9}$$

Setting $x = x_k = 2i \cos(k\pi/n)$ in (9), we see that

$$u(x_k) = i \cos(k\pi/n) + \sin(k\pi/n) = i \exp(-ki\pi/n),$$

and

$$v(x_k) = i \exp(ki\pi/n);$$

thus,

$$f_n(x_k) = i^{n-1} \sin(k\pi)/\sin(k\pi/n) = 0$$

as expected, whereas

$$g_n(x_k) = i^n \cdot 2 \cos(k\pi) = 2i^n(-1)^k;$$

or

$$g_n(x_k) = 2\exp(\tfrac{1}{2}i\pi(n-2k)), \quad k = 1, 2, \ldots, n-1. \tag{10}$$

Substituting this last expression into (9), we see that

$$f_n'(x_k) = \frac{2n\exp(\frac{1}{2}i\pi(n-2k))}{4\sin^2(k\pi/n)},$$

or

$$|f_n'(x_k)| = \frac{n}{2\sin^2(k\pi/n)} \tag{11}$$

Therefore, using (6),

$$|D_n| = \prod_{k=1}^{n-1} n/2\sin^2(k\pi/n),$$

or

$$|D_n| = n^{n-1}\left\{\prod_{k=1}^{n-1} 2\sin^2(k\pi/n)\right\}^{-1}. \tag{12}$$

To evaluate the expression in (12), we set $x = 2i$ in (7). Then,

$$f_n(2i) = \prod_{k=1}^{n-1}(2i)(1 - \cos k\pi/n) = (2i)^{n-1}\prod_{k=1}^{n-1} 2\sin^2(k\pi/2n).$$

Replacing k by $(n-k)$ in the last expression yields

$$f_n(2i) = (2i)^{n-1}\prod_{k=1}^{n-1} 2\cos^2(k\pi/2n).$$

Therefore,

$$(f_n(2i))^2 = (-4)^{n-1}\prod_{k=1}^{n-1}\sin^2(k\pi/n),$$

or

$$(f_n(2i))^2 = (-2)^{n-1}\prod_{k=1}^{n-1} 2\sin^2(k\pi/n). \tag{13}$$

On the other hand, $u(2i) = v(2i) = i$. Using (1),

$$f_n(2i) = \lim_{z\to i}\left(\frac{z^n - i^n}{z - i}\right) = \lim_{z\to i} nz^{n-1} = ni^{n-1}.$$

Thus,

$$(f_n(2i))^2 = n^2(-1)^{n-1}. \tag{14}$$

Comparing (13) and (14) generates the identity:

$$\prod_{k=1}^{n-1} 2\sin^2(k\pi/n) = \frac{n^2}{2^{n-1}}, \quad n = 2, 3, \ldots . \tag{15}$$

Substituting this expression in (12) yields

$$|D_n| = 2^{n-1}n^{n-3}. \tag{16}$$

To obtain the sign of D_n, we consider the expression given in (5). Then,

$$D_n = \prod_{1\leqslant r<s\leqslant n-1}(2i)^2(\cos r\pi/n - \cos s\pi/n)^2;$$

hence,

$$\operatorname{sgn}(D_n) = \prod_{1\leqslant r<s\leqslant n-1}(-1) = \prod_{s=2}^{n-1}\prod_{r=1}^{s-1}(-1) = \prod_{s=2}^{n-1}(-1)^{s-1} = (-1)^{(1+2+\cdots+n-2)},$$

or

$$\operatorname{sgn}(D_n) = (-1)^{\binom{n-1}{2}}.$$

Finally, combining (16) and (17), we obtain

$$D_n = (-1)^{\binom{n-1}{2}} 2^{n-1} n^{n-3}, \quad n \geqslant 3. \tag{18}$$

Note also that setting $n = 2$ in (18) yields the correct expression

$$D_2 = 1.$$

Hence, the proposer's conjecture is correct.

Note: The proposer observed that some results regarding discriminants of Chebyshev polynomials (among others) were discussed in reference [6] of the proposed problem. This reference was unavailable to this solver; it may be shown, however, that the f_n' are, in fact, modified Chebyshev polynomials of the second kind, namely,

$$f_n(x) = (-i)^{n-1} U_{n-1}(ix/2) = |U_{n-1}(ix/2)|.$$

This might lead to an alternative (and briefer) derivation of (18).

Also solved by R. Stanley, who used Chebyshev's polynomials.

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