A FIBONACCI SEQUENCE OF DISTRIBUTIVE LATTICES

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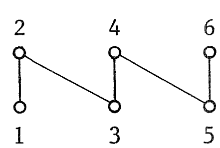
In this paper we describe an order-theoretic realization of the Fibonacci numbers 1, 2, 3, 5, 8, 13, ... and of the Bisection Lucas numbers 3, 7, 18, 47, 123, ... The Bisection Lucas numbers are part of the Lucas sequence and are obtained from the Lucas numbers 2, 1, 3, 4, 7, 11, ... by deleting 2, 1, 4, and then every second number after that. We represent the Fibonacci numbers and the Bisection Lucas numbers as the cardinalities of sequences of distributive lattices that we glue together from simple building blocks. The gluing process is described in Section 2, and the main results are formulated in Section 3 as Theorem 3.1, Theorem 3.4, and their corollaries. In Section 1, we introduce some essential terminology and necessary facts about function lattices. For a more complete treatment of these topics, we refer the reader to the standard textbooks [1], [2], [5], and to [3]. For a related recursive construction of a sequence of modular lattices whose cardinalities are the polygonal numbers, we refer the reader to [6]. It should be noted that the construction discussed in [6] is very different from the construction discussed here in Section 2.

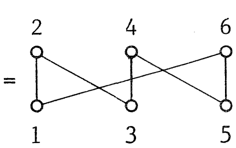
1. FENCES, CROWNS, AND FUNCTION LATTICES

Let P be a partially ordered set, then $|P|$ is the cardinality of P and P^* is the dual of P. For integers $n \geqslant 0$, $\mathbf{n} = \{1, 2, \ldots, n\}$ is the totally ordered chain of n elements ordered in their natural order, $\mathbf{0}$ is the empty chain. The partially ordered set $F(n) = \{i \mid 1 \leqslant i \leqslant n\}$ for $n \geqslant 1$ is a fence if it has the following order:

$$\begin{aligned} i &< i + 1 \quad \text{if } i \text{ is odd,} \\ i &> i + 1 \quad \text{if } i \text{ is even.} \end{aligned} \tag{1.1}$$

From the $2n$-element fence $F(2n)$, for $n \geqslant 2$, we construct the $2n$-element crown $C(2n)$ by introducing exactly one additional order relation, namely $1 < 2n$. For example,

$F(6) =$ 

and $C(6) =$ 

We extend the definitions to include $C(0) = F(0) = \mathbf{0}$ and $C(2) = F(2) = \mathbf{2}$.

For partially ordered sets P, Q, we define Q^P to be the set of all order-preserving mappings $f: P \to Q$ partially ordered by

$$f \leqslant g \text{ if and only if } f(x) \leqslant g(x) \text{ for all } x \in P. \tag{1.2}$$

If f, $g \in Q^P$, then the supremum of f and g, $f \vee g$, exists in Q^P if and only if the supremum of $f(x)$ and $g(x)$ exists in Q for all $x \in P$ and

$$(f \vee g)(x) = f(x) \vee g(x).$$

Since the same is true for the infimum of f and g, it follows that Q^P is a lattice whenever Q is a lattice, P may be an arbitrary partially ordered set. It can be easily verified that Q^P is a distributive or modular lattice, provided that Q is a distributive or modular lattice. All of the partially ordered sets of the form Q^P that we study in this paper are distributive lattices. We are particularly interested in the distributive lattices $\mathbf{2}^{F(n)}$ and $\mathbf{2}^{C(n)}$, for $n \geqslant 0$. Note that $\mathbf{2}^{F(0)} = \mathbf{2}^{C(0)} = \mathbf{1}$, $\mathbf{2}^{F(1)} = \mathbf{2}$, and $\mathbf{2}^{F(2)} = \mathbf{2}^{C(2)} = \mathbf{3}$. As a convenient notation for an order-preserving function $f: F(n) \to \mathbf{2}$, we use its representation by its image vector, i.e., 11212 stands for the function $f: F(5) \to \mathbf{2}$ given by $f(1) = f(2) = f(4) = 1 \in \mathbf{2}$ and $f(3) = f(5) = 2 \in \mathbf{2}$.

A list of arithmetical rules for the exponentiation of arbitrary partially ordered sets P, Q, R may be found in [2] and [3]. We restate here only two that will be needed later.

$$(Q^P)^R \cong Q^{P \times R} \cong (Q^R)^P \tag{1.3}$$

$$(Q^P)^* \cong (Q^*)^{P^*} \tag{1.4}$$

Since we want to recursively construct the lattices $\mathbf{2}^{F(n)}$ and $\mathbf{2}^{C(n)}$ for increasing n, we shall first describe a process of gluing for lattices that is the basis of our recursive construction.

2. A LATTICE CONSTRUCTION

Let L be a lattice. An ideal in L is a nonempty subset $I \subset L$ such that for x, $y \in I$ also $x \vee y \in I$, and for $a \in I$, $x \in L$, $x \leqslant a$ implies $x \in I$. The dual concept is called a filter or a dual ideal in L. Now let L be a lattice and let $I \subset L$ be an ideal. We glue an order-isomorphic copy I' of I below I to L as follows: Let M be the disjoint union of L and I' with the order defined as

$$\begin{aligned} x \leqslant_M y \text{ if any only if } & x \leqslant_L y \\ \text{or } & x \leqslant_{I'} y \\ \text{or } & x = i' < i \leqslant_L y \quad \text{for some } i \in I. \end{aligned} \tag{2.1}$$

With this order M is a lattice where the lattice operations are the given ones on L and on I' and in addition we have $x \vee_M i' = x \vee_L i$ and $x \wedge_M i' = (x \wedge_L i)'$. With this structure, M will be denoted by $L \downarrow I$. Similarly, if $F \subset L$ is a filter, we can glue a copy F' of F above F to the lattice L. The order on the disjoint union K of L and of F' is then defined as

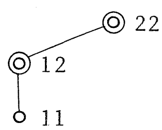
$$\begin{aligned} x \leqslant_K y \text{ if and only if } & x \leqslant_L y \\ \text{or } & x \leqslant_{F'} y \\ \text{or } & x \leqslant_L f < f' = y \quad \text{for some } f \in F, \end{aligned} \tag{2.2}$$

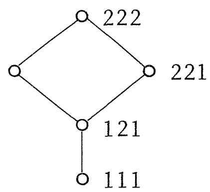
and the lattice operations are defined accordingly. With this structure, K will be denoted as $L \uparrow F$. $L \uparrow F$ and $L \downarrow I$ are distributive or modular lattices whenever L is a distributive or modular lattice, and L is a sublattice of both $L \uparrow F$ and $L \downarrow I$. Moreover, since the gluing constructions are duals of each other, we have the De Morgan properties

$$\begin{aligned}(L \downarrow I)^* &\cong L^* \uparrow I^* \\ (L \uparrow F)^* &\cong L^* \downarrow F^*\end{aligned} \tag{2.3}$$

for any lattice L, ideal $I \subset L$ and filter $F \subset L$.

To illustrate how we will use this construction in the next section, let us look at $\mathbf{2}^{F(2)} \uparrow \mathbf{2}^{F(1)} = \mathbf{3} \uparrow \mathbf{2}$, where the elements of the dual ideal $\mathbf{2}$ in $\mathbf{3}$ are circled:

$\mathbf{2}^{F(2)} = \mathbf{3} =$ 

$\mathbf{3} \uparrow \mathbf{2} =$ 

But the latter is $\mathbf{2}^{F(3)}$ with the mappings indicated in the diagram, so we get that $\mathbf{2}^{F(3)} = \mathbf{2}^{F(2)} \uparrow \mathbf{2}^{F(1)}$.

This construction can, in a rather loose sense, be considered an opposite of a construction used in [4]. In our case, a separate copy of an ideal I or filter F of a lattice L is added to L and the new lattice has cardinality

$$|L| + |I| \quad \text{or} \quad |L| + |F|,$$

whereas in [4] a filter F in a lattice L_1 is identified with an isomorphic ideal I in a lattice L_2 and the new lattice has cardinality

$$|L_1| + |L_2| - |F| = |L_1| + |L_2| - |I|.$$

In both constructions, modularity and distributivity are preserved and the old lattices are sublattices of the new ones.

3. A FIBONACCI SEQUENCE OF DISTRIBUTIVE LATTICES

We are now ready to recursively construct the sequence of distributive lattices whose cardinalities are the Fibonacci numbers.

Theorem 3.1: (1) $\mathbf{2}^{F(n)} \cong \mathbf{2}^{F(n-1)} \downarrow \mathbf{2}^{F(n-2)}$ if n is even, $n \geqslant 2$.

(2) $\mathbf{2}^{F(n)} \cong \mathbf{2}^{F(n-1)} \uparrow \mathbf{2}^{F(n-2)}$ if n is odd, $n \geqslant 2$.

Proof: (1) If $n \geqslant 2$ and even, n is a maximal element in $F(n)$, and the subset A of $\mathbf{2}^{F(n)}$ where n gets mapped to $2 \in \mathbf{2}$ is order-isomorphic to $\mathbf{2}^{F(n-1)}$. In $\mathbf{2}^{F(n-1)}$ we find the set B of all the mappings where $n - 1$ gets mapped to $1 \in \mathbf{2}$. B is an ideal in $\mathbf{2}^{F(n-1)}$ and B is order-isomorphic to $\mathbf{2}^{F(n-2)}$. Therefore, we can define the bijection $\phi : \mathbf{2}^{F(n-1)} \downarrow \mathbf{2}^{F(n-2)} \to \mathbf{2}^{F(n)}$ as follows:

$$\begin{aligned}\phi(f) = g \text{ if and only if } & g|F(n-1) = f \text{ and } g(n) = 2, \text{ if } f \in \mathbf{2}^{F(n-1)} \\ & g|F(n-2) = f \text{ and } g(n-1) = g(n) = 1, \text{ if } f \in \mathbf{2}^{F(n-2)}.\end{aligned}$$

For any $f \in \mathbf{2}^{F(n-2)}$, the extension $\bar{f} \in \mathbf{2}^{F(n)}$ of f defined by $\bar{f}|F(n-2) = f$ and $\bar{f}(n-1) = 1$ and $\bar{f}(n) = 2$ is a direct upper neighbor of $\phi(f)$ in $\mathbf{2}^{F(n)}$; conversely, for each $g, h \in \mathbf{2}^{F(n)}$ with $f = \phi^{-1}(g) \in \mathbf{2}^{F(n-2)}$ and $\phi^{-1}(h) \in \mathbf{2}^{F(n-1)}$ and $g < h$, the extension $\bar{f}$ of f with $\bar{f}(n-1) = 1$ in $\mathbf{2}^{F(n-1)}$ is a direct upper neighbor of

f and $\bar{f} \leqslant \phi^{-1}(h)$ in $2^{F(n-1)}$. Straightforward calculations will complete the proof that ϕ is an order-isomorphism.

(2) For odd n, n is a minimal element in $F(n)$ and we look for the subset $A \subset 2^{F(n)}$ of functions that map n to $1 \in \mathbf{2}$. As in part (1), A is order-isomorphic to $2^{F(n-1)}$, and in $2^{F(n-1)}$ we find the set B of functions that map $n - 1$ to $2 \in \mathbf{2}$. This set B is a filter in $2^{F(n-1)}$. Dualizing the argument of part (1) completes the proof.

Since $2^{F(0)} \cong \mathbf{1}$ and $2^{F(1)} \cong \mathbf{2}$, we have an obvious consequence.

<u>Corollary</u>: The cardinalities of the sequence of distributive lattices $2^{F(n)}$ for increasing $n \geqslant 0$ are the Fibonacci numbers 1, 2, 3, 5, 8, 13, ...

It is possible to give an alternate recursive representation of the lattices $2^{F(n)}$ which uses only the operator $\uparrow$. In essentially the same fashion as in Theorem 3.1 one proves

<u>Theorem 3.2</u>: For any $n \geqslant 2$, $2^{F(n)} \cong A \uparrow 2^{F(n-2)}$, where

$A = (2^{F(n-1)})^*$ if n is even,

and

$A = 2^{F(n-1)}$ if n is odd.

<u>Proof</u>: Let A be the set of all functions that map $1 \in F(n)$ to $1 \in \mathbf{2}$. Then this set is order-isomorphic to $(2^{F(n-1)})^*$. The rest of the proof is as that for Theorem 3.1.

Since $F(2n)$ is a self-dual partially ordered set, every lattice $2^{F(2n)}$, $n \geqslant 0$, is self-dual also. The two theorems, 3.1 and 3.2, and De Morgan's laws (2.3) explain how this self-duality appears in every other step of the recursive construction. Obviously $2^{F(0)} \cong \mathbf{1}$ and $2^{F(2)} \cong \mathbf{3}$ are self-dual and, for $n > 0$, an induction on n establishes

$$\begin{aligned} 2^{F(2n)} &\cong 2^{F(2n-1)} \downarrow 2^{F(2n-1)} \cong (2^{F(2n-1)})^* \uparrow 2^{F(2n-2)} \\ &\cong (2^{F(2n-1)} \downarrow 2^{F(2n-2)})^* \cong (2^{F(2n)})^*. \end{aligned}$$

In fact, this self-duality is a consequence of the following general theorem, which is proved in the same manner.

<u>Theorem 3.3</u>: Let A and B be lattices so that $B \subset A$ is a self-dual ideal of A. The following statements are equivalent:

(1) $A \downarrow B \cong A^* \uparrow B$.

(2) $A \downarrow B$ is self-dual.

Finally, it should be noted that $2^{F(3)}$ is not self-dual.

<u>Theorem 3.4</u>: $2^{C(2n)} \cong 2^{F(2n-1)} \downarrow (2^{F(2n-3)})^*$
$$\cong (2^{F(2n-2)} \uparrow 2^{F(2n-3)}) \downarrow (2^{F(2n-3)})^* \quad \text{for } n \geqslant 2.$$

<u>Proof</u>: The subset A of $2^{C(2n)}$ where the element $2n \in C(2n)$ gets mapped onto $2 \in \mathbf{2}$ is order-isomorphic to $2^{F(2n-1)}$. In $2^{F(2n-1)}$ we find the set B of all those mappings where 1 and also $2n-1$ get mapped onto $1 \in \mathbf{2}$. B is an ideal in

$\mathbf{2}^{F(2n-1)}$, and it is order-isomorphic to the dual of $\mathbf{2}^{F(2n-3)}$ by (1.4). All maps f in B can be extended to maps $\bar{f}: C(2n) \to \mathbf{2}$ by defining

$$\bar{f}(2n) = 1 \quad \text{and} \quad f|F(2n-1) = f.$$

These are the direct lower neighbors of the maps that have the same images on $F(2n-1)$ but map $2n$ to 2. Clearly, $\mathbf{2}^{C(2n)}$ is the disjoint union of A and an order-isomorphic copy of B, and its order structure is that of

$$\mathbf{2}^{F(2n-1)} \downarrow (\mathbf{2}^{F(2n-3)})^*.$$

For the cardinalities of the lattices $\mathbf{2}^{C(2n)}$, we have

$$|\mathbf{2}^{C(2n)}| = |\mathbf{2}^{F(2n-1)}| + |\mathbf{2}^{F(2n-3)}|, \tag{3.1}$$

and we know already that $|\mathbf{2}^{F(n)}|$ for $n \geqslant 0$ are the Fibonacci numbers. The sum of the n^{th} and the $(n+2)^{\text{nd}}$ Fibonacci numbers generates another Fibonacci sequence which is part of the Lucas sequence 2, 1, 3, 4, 7, 11, ... From the Lucas sequence, the Bisection Lucas sequence ([7], p. 101, #1067) is generated by deleting 2, 1, 4, and every second number after that. Since $|\mathbf{2}^{C(2)}| = 3$, and because of (3.1), we have the following

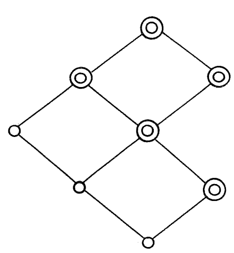
Corollary: The cardinalities of the sequence of distributive lattices $\mathbf{2}^{C(2n)}$ for increasing $n \geqslant 1$ are the Bisection Lucas numbers 3, 7, 18, 47, 123, ...

For an interesting extension of the corollaries to Theorem 3.1 and Theorem 3.4, we replace the two-element chain in the base of our function lattices by the Boolean algebra $\mathbf{2}^k$, $k \geqslant 1$ denoting a k-element antichain. Then, $(\mathbf{2}^k)^{F(n)} \cong (\mathbf{2}^{F(n)})^k$ by (1.3) and, therefore, we have as a consequence of the corollary to Theorem 3.1 that the cardinalities of $(\mathbf{2}^k)^{F(n)}$ for $n \geqslant 0$ are given by the k^{th} powers of the Fibonacci numbers, $1^k, 2^k, 3^k, 5^k, 8^k, \ldots$. Similarly, $(\mathbf{2}^k)^{C(2n)} \cong (\mathbf{2}^{C(2n)})^k$ and, as a consequence of the corollary to Theorem 3.4, the cardinalities of $(\mathbf{2}^k)^{C(2n)}$, $n \geqslant 0$, are the k^{th} powers of the Bisection Lucas numbers, $3^k, 7^k, 18^k, 47^k, \ldots$.

We conclude the paper with an example which illustrates our construction. We show that our method of gluing provides a completely symmetrical construction of the free distributive lattice on three generators, that is, the lattice $\mathbf{2}^{C(6)}$ which has 18 elements. We construct $\mathbf{2}^{C(6)}$ as follows:

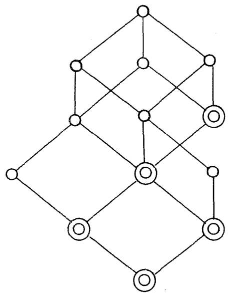
$$\mathbf{2}^{C(6)} \cong \mathbf{2}^{F(5)} \downarrow (\mathbf{2}^{F(3)})^* \cong (\mathbf{2}^{F(4)} \uparrow \mathbf{2}^{F(3)}) \downarrow (\mathbf{2}^{F(3)})^*.$$

The circled elements in the figure below are those of the filter $\mathbf{2}^{F(3)}$ in $\mathbf{2}^{F(4)}$, consisting of the maps where $4 \in F(4)$ is mapped to $2 \in \mathbf{2}$.

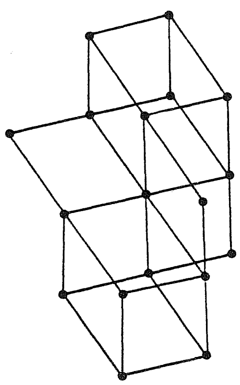


$$= \mathbf{2}^{F(4)} = \mathbf{2}^{F(3)} \downarrow \mathbf{2}^{F(2)}$$

To get $\mathbf{2}^{F(5)} \cong \mathbf{2}^{F(4)} \uparrow \mathbf{2}^{F(3)}$, we glue a copy of $\mathbf{2}^{F(3)}$ above $\mathbf{2}^{F(3)}$ as shown in the following figure. Here the mappings where 1 and 5 in $F(5)$ both go to $2 \in \mathbf{2}$ are circled. This circled set is an ideal in $\mathbf{2}^{F(5)}$ and it is an isomorphic copy of the dual of $\mathbf{2}^{F(3)}$.



Finally, we attach a copy of the circled ideal in the figure for $\mathbf{2}^{F(5)}$ and get the free distributive lattice $\mathbf{2}^{C(6)}$.



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