INTEGERS RELATED TO THE BESSEL FUNCTION $J_1(z)$

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1. INTRODUCTION

Let $J_\nu(z)$ denote the Bessel function of the first kind and let $j_{\nu,r}$ denote the zeros of $z^{-\nu}J_\nu(z)$, with $|R(j_{\nu,r})| \leqslant |R(j_{\nu,r+1})|$. The Rayleigh function of order $2n$, $\sigma_{2n}(\nu)$, is defined by

$$\sigma_{2n}(\nu) = \sum_{r=1}^{\infty} (j_{\nu,r})^{-2n} \qquad (n = 1, 2, 3, \ldots).$$

The early history of this function can be found in [10, p. 502]; more recently it has been investigated by Kishore [5], [6] and others. The first twelve Rayleight functions have been computed by Lehmer [8].

It is known that

$$\sigma_{2n}(1/2) = (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} B_{2n},$$

$$\sigma_{2n}(-1/2) = (-1)^{n} \frac{2^{2n-2}}{(2n)!} G_{2n},$$

where B_{2n} is the $2n^{\text{th}}$ Bernoulli number and G_{2n} is the Genocchi number, i.e.,

$$G_{2n} = 2(1 - 2^{2n})B_{2n}.$$

A few other special cases have been examined. The writer [2], [3], and [4] has studied the cases $\nu = \pm 3/2$ and Carlitz [1] has investigated the integers a_r defined by

$$\sigma_{2r}(0) = \frac{2^{-2r}}{r!(r-1)!} a_r. \tag{1.1}$$

Carlitz points out that in view of the known arithmetic properties of the Bernoulli and Genocchi numbers, it is of interest to look for arithmetic properties of $\sigma_{2n}(\nu)$ for other values of ν.

In the present paper we define integers b_r by means of

$$\sigma_{2r}(1) = \frac{2^{-2r}}{r!(r+1)!} b_r, \tag{1.2}$$

and examine their arithmetic properties. A summary of these properties, along with a possible generalization of (1.1) and (1.2), is given in Section 4. A listing of the first 24 values of b_n is presented in section 5.

2. PRELIMINARIES

Using formulas (6), (14), and (22) in [5], we can write a generating function and recurrence formulas for b_n. We have

$$\frac{-x}{2}\frac{J_1'(x)}{J_1(x)} + \frac{1}{2} = \sum_{n=1}^{\infty} \frac{2^{-2n}}{n!(n+1)!} b_n x^{2n}, \tag{2.1}$$

$$(-1)^n (n+1) b_n = -n(n+1) + \sum_{r=1}^{n-1} (-1)^{r-1} \binom{n+1}{r+1}\binom{n+1}{r} b_r, \tag{2.2}$$

$$(n+1)^2 b_n = \sum_{r=1}^{n-1} \binom{n+1}{r+1}\binom{n+1}{r} b_r b_{n-r}. \tag{2.3}$$

It follows from (1.2) that $b_1 = 1$, $b_2 = 1$, $b_3 = 3$, $b_4 = 16$. In some of our proofs it will be convenient to rewrite (2.2) in the following way:

$$(-1)^n (n+1) b_n = -n(n+1) + \sum_{r=1}^{n-1} A(n, r), \tag{2.4}$$

where

$$A(n, r) = (-1)^{r-1}\binom{n+1}{r+1}\binom{n+1}{r} b_r.$$

To derive properties of b_n from (2.2) and (2.3) we need the following lemmas, the first due to Lucas [9] and the second due to Kummer [7]. In Lemma 2.2, and throughout this paper, we use the notation $p^m \| h$ to mean $p^m | h$ and $p^{m+1} \nmid h$.

Lemma 2.1: If p is a prime number and

$$n = n_0 + n_1 p + \cdots + n_k p^k \qquad (0 \leqslant n_i < p)$$
$$r = r_0 + r_1 p + \cdots + r_k p^k \qquad (0 \leqslant r_i < p),$$

then

$$\binom{n}{r} \equiv \binom{n_0}{r_0}\binom{n_1}{r_1} \cdots \binom{n_k}{r_k} \pmod{p}.$$

Lemma 2.2: With the hypotheses of Lemma 2.1, let $n - r = s_0 + s_1 p + \cdots + s_k p^k$ with $0 \leqslant s_i < p$, and suppose

$$\begin{aligned} r_0 + s_0 &= u_0 p + c_0 \qquad (0 \leqslant c_0 < p) \\ u_0 + r_1 + s_1 &= u_1 p + c_1 \qquad (0 \leqslant c_1 < p) \\ &\vdots \\ u_{k-1} + r_k + s_k &= u_k p + c_k \qquad (0 \leqslant c_k < p). \end{aligned}$$

Then

$$p^N \bigg\| \binom{n}{r}, \text{ where } N = u_0 + u_1 + \cdots + u_k.$$

It follows from Lemma 2.2 that, if $r_j > n_j$ and $r_{j+t} \geqslant n_{j+t}$ for $t = 1, \ldots, q - 1$, then

$$\binom{n}{r} \equiv 0 \pmod{p^q}.$$

It may be of interest to note the following relationship between the numbers defined by (1.1) and (1.2). This formula follows easily from Eq. (20) in [5]: for $n > 1$,

$$na_n = \sum_{r=1}^{n-1} \binom{n}{r}\binom{n}{r+1} b_r a_{n-r}.$$

3. PROPERTIES OF b_n

Since

$$\binom{n+1}{r+1}\binom{n+1}{r}\Big/(n+1)$$

is always an integer, it is evident from (2.2) that the b_n are positive integers. Our first five theorems are concerned with determining the prime factors of b_n.

Theorem 3.1: Let $n = 2^k m$, $k \geqslant 0$, m odd. Then $b_n \equiv 0 \pmod{m}$.

Proof: The proof is by induction on n. Using the table in Section 5, we can verify the theorem for $n = 1, 2, \ldots, 24$. Assume it is true for $n = 1, \ldots, j - 1$ and suppose $p^s \| j$, $p > 2$. In (2.4) replace n by j and suppose $p^t \| r$ for a fixed r. If $s < t$, then $b_r \equiv 0 \pmod{p^s}$ by the induction hypothesis. If $0 < t < s$, then

$$b_r \equiv 0 \pmod{p^t} \quad \text{and} \quad \binom{j+1}{r+1} \equiv 0 \pmod{p^{s-t}} \quad \text{by Lemma 2.2.}$$

If $t = 0$, then

$$\text{either } \binom{j+1}{r} \equiv 0 \pmod{p^s} \quad \text{or} \quad \binom{j+1}{r+1} \equiv 0 \pmod{p^s} \quad \text{by Lemma 2.2.}$$

In all cases, $A(j, r) \equiv 0 \pmod{p^s}$, and by (2.4) we see that $b_j \equiv 0 \pmod{p^s}$. This completes the proof.

It follows that if p is an odd prime then $b_p \equiv 0 \pmod{p}$. Also, if we replace n by $p - 1$ in (2.2) and observe that

$$\binom{p}{r+1}\binom{p}{r} \equiv 0 \pmod{p^2} \text{ for } r = 1, \ldots, p - 2,$$

we have

$$b_{p-1} \equiv 1 \pmod{p}, \tag{3.1}$$

where p is an odd prime. The next two theorems give more results along this line.

Theorem 3.2: Let p be an odd prime and $0 \leqslant k < p - 2$. Then $b_{mp+k} \equiv 0 \pmod{p}$ for all $m \geqslant 1$.

Proof: We first show the theorem is true for $m = 1$. It is true for $m = 1$, $k = 0$, by Theorem 3.1. Assume it is true for $m = 1$ and $k = 0, \ldots, j - 1$, with $j < p - 2$. Then by (2.4) and Lemma 2.1, we have

$$\begin{aligned}(-1)^{p+j}(p + j + 1)b_{p+j} &= -(p + j)(p + j + 1) + \sum_{r=1}^{p+j-1} A(p + j, r)\\ &\equiv -j(j + 1) + \sum_{r=1}^{j} A(p + j, r) \pmod{p}\\ &\equiv -j(j + 1) + \sum_{r=1}^{j} A(j, r) \pmod{p} \equiv 0 \pmod{p},\end{aligned}$$

the last congruence following from (2.4). Thus, the theorem is true for $m = 1$. Now assume it is true for $m = 1, \ldots, h - 1$. We know $b_{hp} \equiv 0 \pmod p$ by Theorem 3.1, so we also assume the theorem is true for $m = h$ and $k = 0, \ldots, j - 1$, with $j < p - 2$. Then, as in the first part of the proof, we have

$$(-1)^{hp+j}(hp + j + 1)b_{hp+j} \equiv -j(j + 1) + \sum_{r=1}^{j} A(j, r) \equiv 0 \pmod p,$$

which completes the proof.

Theorem 3.2 tells us that if $n > p - 1$ and $n \not\equiv -1$, $n \not\equiv -2 \pmod p$, then $b_n \equiv 0 \pmod p$. The cases $n \equiv -1$, $n \equiv -2 \pmod p$ are examined in the following theorem.

Theorem 3.3: Let p be an odd prime. Then for all $m > 1$, $b_{mp-1} \equiv b_{mp-2} \equiv a_m \pmod p$, where a_m is defined by (1.1).

Proof: In (2.2), we replace n by $mp - 1$ and divide out p. Then, by Lemma 2.1, Lemma 2.2, and Theorem 3.2,

$$(-1)^{mp-1}b_{mp-1} \equiv 1 + \sum_{r=1}^{m-1}(-1)^r\binom{m}{r}\binom{m-1}{r-1}b_{rp-1} \pmod p,$$

with $b_{p-1} \equiv 1 \pmod p$. In [1] it is shown that $a_1 = 1$ and

$$(-1)^{m-1}a_m = 1 + \sum_{r=1}^{m-1}(-1)^r\binom{m}{r}\binom{m-1}{r-1}a_r. \tag{3.2}$$

It follows that $b_{mp-1} \equiv a_m \pmod p$. Now, in (2.2), replace n by $mp - 2$. Then we have

$$(-1)^{m-1}b_{mp-2} \equiv -2 + \sum_{r=1}^{p-2}A(p - 2, r) + \sum_{r=2}^{m-1}(-1)^r\binom{m-1}{r-1}^2 b_{rp-2}$$
$$+ \sum_{r=1}^{m-1}(-1)^r\binom{m-1}{r}\binom{m-1}{r-1}b_{rp-1} \pmod p. \tag{3.3}$$

Note that $-2 + \sum A(p - 2, r) \equiv 0$ by (2.4). We see from (3.3) that

$$b_{2p-2} \equiv 1 \equiv a_2 \equiv b_{2p-1} \pmod p;$$

we now proceed to show $b_{mp-2} \equiv a \pmod p$ by using induction on m in (3.3). If Theorem 3.3 is true for $m = 2, \ldots, j - 1$, then by (3.3) we have

$$(-1)^{j-1}b_{jp-2} \equiv \sum_{r=2}^{j-1}(-1)^r a_r\binom{j-1}{r-1}\left[\binom{j-1}{r-1} + \binom{j-1}{r}\right] - j + 1$$
$$\equiv 1 + \sum_{r=1}^{j-1}(-1)^r\binom{j}{r}\binom{j-1}{r-1}a_r \equiv a_j \pmod p.$$

This completes the proof of Theorem 3.3.

Carlitz [1] has shown that, if $n = mp^r$, then $a_n \equiv a_m \pmod p$ for $r = 0, 1, 2, \ldots$. Therefore, we have the following corollary.

Corollary: If p is an odd prime and $n = mp^r - 1$ or $n = mp^r - 2$, then $b_n \equiv a_m \pmod p$ for $r = 1, 2, 3, \ldots$.

It follows from the corollary that, if $m > p$ and $p \nmid m$, then $b_n \equiv 0 \pmod{p}$ for $n = mp^r - 1$ or $n = mp^r - 2$.

We next show that Theorem 3.3 is valid for $p = 2$.

Theorem 3.4: For $m \geqslant 1$, $b_{2m+1} \equiv b_{2m} \equiv a_{m+1} \pmod{2}$.

Proof: We first show that $b_{4m} \equiv 0 \pmod{2}$ for all $m \geqslant 1$. It is clear from Lemma 2.1 that

$$\binom{4m+1}{r}\binom{4m+1}{r+1} \equiv 0 \pmod{2} \text{ for } r \equiv 1, 2, \text{ or } 3 \pmod{4}.$$

Therefore, by (2.2), we have

$$b_{4m} \equiv \sum_{r=1}^{m-1} \binom{4m+1}{4r}\binom{4m+1}{4r+1} b_{4r} \pmod{2}.$$

Since $b_4 = 16$, we can now easily prove by induction that $b_{4m} \equiv 0 \pmod{2}$. Now we replace n by $2m + 1$ in (2.2) and divide out $2m + 2$. Then we have

$$\begin{aligned} b_{2m+1} &\equiv 1 + \sum_{r=1}^{m}\binom{m}{r}\binom{m+1}{r} b_{2r} + \sum_{r=1}^{m}\binom{m+1}{r}\binom{m}{r-1} b_{2r-1} \\ &\equiv 1 + \sum_{r=1}^{m}\binom{m+1}{r}\binom{m}{r-1} b_{2r-1} \pmod{2}, \end{aligned}$$

because $b_{4k} \equiv 0 \pmod{2}$ and because

$$\binom{m}{r}\binom{m+1}{r} \equiv 0 \pmod{2} \text{ if } r \text{ is odd.}$$

Since $b_1 = 1$, we now see by (3.2) that $b_{2m+1} \equiv a_{m+1} \pmod{2}$.

Next assume that $b_{2m} \equiv a_{m+1} \pmod{2}$ for $m = 1, \ldots, j - 1$. Replace n by $2j$ in (2.2) to obtain

$$b_{2j} \equiv \sum_{r=1}^{j-1}\binom{j}{r}^2 a_{r+1} + \sum_{r=1}^{j}\binom{j}{r}\binom{j}{r-1} a_r \equiv -1 + \sum_{r=1}^{j}\binom{j+1}{r}\binom{j}{r-1} a_r \pmod{2}.$$

By (3.2), we now have $b_{2j} \equiv a_{j+1} \pmod{2}$, which completes the proof.

It follows that, if $n = 2^k - 1$ or $n = 2^k - 2$, then b_n is odd, $k = 1, 2, 3, \ldots$. Otherwise b_n is even. These facts enable us to extend Theorem 3.1.

Theorem 3.5: $b_n \equiv 0 \pmod{n}$ unless $n = 2^j$, $j = 2, 3, \ldots$. If $n = 2^j - 2$, then $b_n \equiv 0 \pmod{n/2}$.

Proof: We use induction on n. Theorem 3.5 is valid for $n = 1, 2, \ldots, 24$; assume it is true for $n = 1, \ldots, k - 1$. We assume k is even and $k \neq 2^j - 2$, since otherwise, by Theorem 3.1, there is nothing to prove. Assume $2^s \| k$ and $2^t \| r$ for a fixed r, $1 \leqslant r \leqslant k - 1$. If $t > s$, then $b_r \equiv 0 \pmod{2^s}$ by induction hypothesis, and $A(k, r) \equiv 0 \pmod{2^s}$. If $1 < t < s$, then

$$\binom{k+1}{r+1}\binom{k+1}{r} \equiv 0 \pmod{2^{2s-2t}}$$

and $b_r \equiv 0 \pmod{2^t}$, so $A(k, r) \equiv 0 \pmod{2^s}$. If $1 < t < s$, then

$$\binom{k+1}{r+1}\binom{k+1}{r} \equiv 0 \pmod{2^{2s-2}},$$

and $A(k, r) \equiv 0 \pmod{2^s}$. Thus, if $t > 0$ and $s > 1$, $A(k, r) \equiv 0 \pmod{2^s}$. It is now easy to see that, if $s > 1$, we have, by (2.4) and Lemma 2.2,

$$b_k \equiv A(k, 1) + A(k, 2^s - 1) \equiv 2^{s-1} + 2^{s-1} \equiv 0 \pmod{2^s}.$$

If $s = 1$, let $2^{m+1} \| (k+2)$, $m \geqslant 1$. Then by (2.4),

$$b_k \equiv \sum_{i=1}^{m} A(k, 2^i - 1) + \sum_{i=2}^{m+1} A(k, 2^i - 2) \equiv 2m \equiv 0 \pmod 2,$$

and the proof is complete.

If we replace n by an odd prime in (2.2), then since

$$\binom{p+1}{r+1}\binom{p+1}{r} \equiv 0 \pmod{p^2} \text{ for } r = 2, \ldots, p-2,$$

it is easy to see that

$$b_p \equiv p \pmod{p^2}. \tag{3.4}$$

In the same way, we can show that if $p > 3$, then

$$b_{p+1} \equiv \frac{7}{6}p \pmod{p^2}. \tag{3.5}$$

If we set $b_{p+n} \equiv pd_n \pmod{p^2}$, we can find a simple generating function for d_n.

Theorem 3.6: Let p be an odd prime and let $0 \leqslant n \leqslant p-3$. Then $b_{p+n} \equiv pd_n$ (mod p^2), where

$$1 + \sum_{n=0}^{\infty} \frac{d_n(x/2)^{2n+2}}{n!(n+1)!} = \left(\frac{x}{2J_1(x)}\right)^2.$$

Proof: Define $d_n^{(p)}$ by $b_{p+n} \equiv pd_n^{(p)} \pmod{p^2}$ for $0 \leqslant n \leqslant p-3$, and replace n by $p+n$ in (2.3). Using Lemma 2.1, we see that $d_n^{(p)} \equiv d_n \pmod p$, where

$$(n+1)^2 d_n = 2\sum_{r=1}^{n} \binom{n+1}{r+1}\binom{n+1}{r} b_r d_{n-r} + \frac{2b_{n+1}}{n+2} \tag{3.6}$$

with $d_0 = 1$. We multiply both sides of (3.6) by $(x/2)^{2n+2}$ and sum, beginning at $n = 0$, to obtain

$$\frac{x}{2} D'(x) = 2B(x)D(x), \tag{3.7}$$

where

$$D(x) = 1 + \sum_{n=0}^{\infty} \frac{d_n(x/2)^{2n+2}}{n!(n+1)!},$$

$$B(x) = \sum_{n=1}^{\infty} \frac{b_n(x/2)^{2n}}{n!(n+1)!} = -\frac{x}{2}\,\frac{J'(x)}{J\ (x)} + \frac{1}{2},$$

the last equation following from (2.1). Thus,

$$\frac{D'(x)}{D(x)} = -2\,\frac{J_1'(x)}{J_1(x)} + \frac{2}{x}. \tag{3.8}$$

After integrating both sides of (3.8) and plugging in $x = 0$ to determine the constant, we have

$$D(x) = \left(\frac{x}{2J_1(x)}\right)^2,$$

which completes the proof.

Theorem 3.5 can be compared to a similar result for the a_n. Carlitz [1] has shown that for $1 \leqslant n < p$, $a_{p+n} \equiv c_n p \pmod{p^2}$, where the c_n are defined by

$$1 + \sum_{n=1}^{\infty} \frac{c_n (x/2)^{2n}}{(n-1)!(n-1)!} = (J_0(x))^{-2}.$$

Theorem 3.7: If p is a prime number and $n = p^s$, $s \geqslant 3$, then $b_n \equiv p^s \pmod{p^{s+1}}$. If p is odd, the congruence is valid for $s \geqslant 1$.

Proof: First, assume p is odd. Theorem 3.1 tells us that, if $p^t | r$, then $b_r \equiv 0 \pmod{p^t}$; we also note that, if $j = p^s - 1$, then $b_j \equiv 1 \pmod{p}$ by the corollary to Theorem 3.3. Now, in (2.4), replace n by p^s. It is clear from Lemma 2.2 and the above comments that $A(p^s, r) \equiv 0 \pmod{p^{s+1}}$ for $r = 2, \ldots, p^s - 2$. We therefore have, for $n = p^s$,

$$\begin{aligned}(p^s + 1)b_n &\equiv (p^s + 1)p^s + A(p^s, 1) + A(p^s, p^s - 1)\\ &\equiv (p^s + 1)p^s \pmod{p^{s+1}}.\end{aligned}$$

This proof is valid for $s \geqslant 1$.

For $p = 2$, the situation is more complicated. We first show that, if $m = 2^s - 1$ with $s > 2$, then $b_m \equiv 1 \pmod{4}$. In (2.4), replace n by $2^s - 1$, $s > 2$. It is easy to see by Lemma 2.2 and Theorem 3.5 that $A(2^s - 1, r) \equiv 0 \pmod{2^{s+2}}$ for each r except $r = 2^{s-1} - 1$; in that case, $A(2^s - 1, 2^{s-1} - 1) \equiv 0 \pmod{2^{s+1}}$. After dividing both sides of (2.4) by 2 , we have, for $m = 2^s - 1$,

$$b_m \equiv -1 + A(2^s - 1, 2^{s-1} - 1)/2^s \equiv -1 + 2 \equiv 1 \pmod{4}.$$

Now, replace n by 2^s in (2.4). For $r = 1, \ldots, 2^s - 1$, it is easy to see, by Lemma 2.2 and Theorem 3.5, that $A(2^s, r) \equiv 0 \pmod{2^{s+1}}$ if $2^t \| r$ with $t \geqslant 1$. If $t = 0$, then $A(2^s, r) \equiv 0 \pmod{2^{s+1}}$ except for $r = 1$, $2^s - 1$, and $2^{s-1} - 1$. We therefore have, by (2.4) with $w = 2^s$,

$$\begin{aligned}b_w &\equiv 2^s + A(2^s, 1) + A(2^s, 2^s - 1) + A(2^s, 2^{s-1} - 1)\\ &\equiv 2^s + 2^{s-1} + 2^{s-1} + 2^s \equiv 2^s \pmod{2^{s+1}}.\end{aligned}$$

4. SUMMARY

We have shown that the integers b_n defined by (1.2) have the following properties:

$b_n \equiv 0 \pmod{n}$ unless $n = 2^j - 2$, $j = 2, 3, \ldots$. If $n = 2^j - 2$, then $b_n \equiv 0 \pmod{n/2}$. (4.1)

$b_{mp+k} \equiv 0 \pmod{p}$ if p is an odd prime, $0 \leqslant k \leqslant p - 3$, and $m \geqslant 1$. (4.2)

$b_{mp-1} \equiv b_{mp-2} \equiv a_m \pmod{p}$ if p is any prime number, $m > 1$ and a_m is defined by (1.1). (4.3)

$b_{p+n} \equiv pd_n \pmod{p^2}$, if p is an odd prime, $0 \leqslant n \leqslant p - 3$, and d_n is defined by

$$1 + \sum_{n=0}^{\infty} \frac{d_n(x/2)^{2n+2}}{n!\,(n+1)!} = \left(\frac{x}{2J_1(x)}\right)^2. \tag{4.4}$$

$b_n \equiv p^s \pmod{p^{s+1}}$ if $n = p^s$, p any prime number, and $s \geqslant 3$.
If p is odd, the congruence is valid for $s \geqslant 1$. (4.5)

To generalize (1.1) and (1.2), we can define the numbers $a_{k,n}$ by

$$\sigma_{2n}(k) = \frac{2^{-2n}}{(n+k)!\,(n+k-1)!}\, a_{k,n}.$$

It is evident that $a_{0,n} = a_n$ and $a_{1,n} = b_n$. Also, $a_{k,1} = a_{k,2} = (k!)^2$. Formulas analogous to (2.1), (2.2), and (2.3) can be written down, but properties such as (4.1)-(4.5) do not appear to be obvious or easily proved.

5. TABLE OF VALUES

The following table of values for b_n was computed by Elmer Hayashi of Wake Forest University. The writer is grateful to Professor Hayashi for his assistance. The writer also wishes to thank John Baxley of Wake Forest and Sam Wagstaff of Purdue University for their help in proving that all the factors listed below are prime numbers.

Table of Values for b_n

$b_1 = 1$
$b_2 = 1$
$b_3 = 3$
$b_4 = 2^4$
$b_5 = 2 \cdot 5 \cdot 13$
$b_6 = 3^3 \cdot 5 \cdot 11$
$b_7 = 5 \cdot 7 \cdot 647$
$b_8 = 2^3 \cdot 7^2 \cdot 11 \cdot 103$
$b_9 = 2^2 \cdot 3^2 \cdot 7 \cdot 79 \cdot 547$
$b_{10} = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 777013$
$b_{11} = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 195407$
$b_{12} = 2^5 \cdot 3^2 \cdot 5 \cdot 11 \cdot 163 \cdot 193189$
$b_{13} = 2^2 \cdot 3 \cdot 11 \cdot 13 \cdot 449 \cdot 1229 \cdot 26119$
$b_{14} = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 677 \cdot 15473 \cdot 44983$
$b_{15} = 3^2 \cdot 5 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 2897 \cdot 9208057$
$b_{16} = 2^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 85619815212829$
$b_{17} = 2^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 263 \cdot 331 \cdot 379 \cdot 25452443$
$b_{18} = 2^3 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17 \cdot 181 \cdot 827 \cdot 22338511427$
$b_{19} = 2^3 \cdot 3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 4974009342476711903$
$b_{20} = 2^5 \cdot 3 \cdot 5 \cdot 13 \cdot 17 \cdot 19 \cdot 137 \cdot 315195497 \cdot 7249259477$
$b_{21} = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19^2 \cdot 395001666315568761311$
$b_{22} = 2^2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19 \cdot 463 \cdot 13394141029047928133$
$b_{23} = 2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 47 \cdot 151 \cdot 60443 \cdot 3308491075235249$
$b_{24} = 2^5 \cdot 3^3 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 23 \cdot 24917 \cdot 21854261271093057456989$

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