DEFINITIONS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$  

Also, $\alpha$ and $\beta$ designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $\alpha^2 - \alpha - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-556 Proposed by Valentina Bakinova, Rondout Valley, NY

State and prove the general result illustrated by

$$4^2 = 16, \quad 34^2 = 1156, \quad 334^2 = 111556, \quad 3334^2 = 11115556.$$  

B-557 Proposed by Imre Mereňyi, Cluj, Romania

Prove that there is no integer $n \geq 2$ such that

$$F_{3n-6}F_{3n-3}F_{3n+3}F_{3n+6} - F_{n-2}F_nF_{n+3} = 1985^8 + 1.$$  

B-558 Proposed by Imre Mereňyi, Cluj, Romania

Prove that there are no positive integers $m$ and $n$ such that

$$F_{4m}F_{3n} - 4 = 0.$$  

B-559 Proposed by László Cseh, Cluj, Romania

Let $\alpha = (1 + \sqrt{5})/2$. For positive integers $n$, prove that

$$[\alpha + .5] + [\alpha^2 + .5] + \cdots + [\alpha^n + .5] = L_{n+2} - 2,$$

where $[x]$ denotes the greatest integer in $x$.  

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ELEMENTARY PROBLEMS AND SOLUTIONS

B-560 Proposed by László Cseh, Cluj, Romania

Let \( a \) and \([x]\) be as in B-559. Prove that
\[
[aF_1 + .5] + [a^2F_2 + .5] + \cdots + [a^nF_n + .5]
\]
is always a Fibonacci number.

B-561 Proposed by Piero Filipponi, Fdn. U. Bordoni, Rome, Italy

(i) Let \( Q \) be the matrix \((1 1)\). For all integers \( n \), show that
\[
Q^n = (-1)^n Q^{-n} = L_n I, \text{ where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(ii) Find a square root of \( Q \), i.e., a matrix \( A \) with \( A^2 = Q \).

SOLUTIONS

Double Product of 4 Consecutive Fibonacci Numbers

B-532 Proposed by Herta T. Freitag, Roanoke, VA

Find \( a, b, \) and \( c \) in terms of \( n \) so that
\[
a^3(b - c) + b^3(c - a) + c^3(a - b) = 2F_n F_{n+1} F_{n+2} F_{n+3}.
\]

Solution by Graham Lord, Princeton, NJ

The cyclic expression on the left-hand side factors into
\[-(a - b)(b - c)(c - a)(a + b + c).
\]
The equality is quickly verified when \( a = F_{n+1}, b = F_{n+2}, \) and \( c = F_{n+3} \).


Product of 5 Fibonacci Numbers

B-533 Proposed by Herta T. Freitag, Roanoke, VA

Let
\[g(a, b, c) = a^n(b^2 - c^2) + b^n(c^2 - a^2) + c^n(a^2 - b^2).\]

Determine an infinitude of choices for \( a, b, \) and \( c \) such that \( g(a, b, c) \) is the product of five Fibonacci numbers.

Solution by Graham Lord, Princeton, NJ

The cyclic expression on the left-hand side factors into
\[-(b^2 - c^2)(c^2 - a^2)(a^2 - b^2).
\]

With \( a = F_n, b = F_{n+1}, \) and \( c = F_{n+2}, \) this becomes \( F_{n-1} F_n F_{n+2} F_{n+3} F_{2n}.\)

ELEMENTARY PROBLEMS AND SOLUTIONS

No Pythagorean Triangle with Square Area

B-534 Proposed by A. B. Patel, India

One obtains the lengths of the sides of a Pythagorean triangle by letting
\[ a = u^2 - v^2, \quad b = 2uv, \quad c = u^2 + v^2, \]
where \( u \) and \( v \) are integers with \( u > v > 0 \). Prove that the area of such a triangle is not a perfect square when \( u = F_{n+1}, \quad v = F_n, \) and \( n \geq 2 \).

I. Solution by L. A. G. Dresel, University of Reading, England

We have
\[ A = \frac{1}{2}ab = F_{n-1}F_nF_{n+1}F_{n+2}. \]
and the area is given by
\[ F_{n-1}F_{n+2} = F_n^2 + (-1)^n. \]

It follows that the area \( A \) is the product of two consecutive integers, and thus cannot be a perfect square if \( F_{n-1} > 0 \), i.e., \( n \geq 2 \). In fact,
\[ A = a(a + 1) \quad \text{when} \ n \text{ is odd}, \]
and
\[ A = a(a - 1) \quad \text{when} \ n \text{ is even}. \]

II. Solution by L. Cseh (Cluj, Romania), J. M. Metzger (Grand Forks, ND), Bob Prielipp (Oshkosh, WI), Sahib Singh (Clarion, PA), and Lawrence Somer (Washington, D.C.), independently.

It is well known that the area of a Pythagorean triangle with integral sides is never a perfect square. For proof, see page 256 of Elementary Number Theory by David M. Burton (1980 edition). Thus, this result is true, in general, and therefore independent of involvement of Fibonacci numbers.

Also solved by Paul S. Bruckman, Piero Filipponi, Walther Janous, K. Klauser, L. Kuipers, I. Merényi, M. Wachtel, Tad White, and the proposer.

Impossible Sum

B-535 Proposed by L. Cseh & I. Merényi, Cluj, Romania

Prove that there is no positive integer \( n \) for which
\[ F_1 + F_2 + F_3 + \cdots + F_{3n} = 16! \]

Solution by L. A. G. Dresel, University of Reading, England

We have the identity
\[ F_1 + F_2 + F_3 + \cdots + F_{3n} = F_{3n+2} - 1, \]
and it remains to prove that there is no integer \( n \) for which \( F_{3n+2} - 1 = 16! \). If there were such an integer \( n \), then, since Wilson’s theorem gives
\[ 16! \equiv -1 \pmod{17} \]
we would require
\[ F_{3n+2} \equiv 0 \pmod{17}. \]
Now, the first Fibonacci number divisible by 17 is \( F_9 = 34 \), and therefore \( F_m \)

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is divisible by 17 if and only if 9 divides \( m \). Clearly, there is no integer \( n \) for which \( 3n + 2 \) is divisible by 9, and the result follows.

Also solved by Paul S. Bruckman, Piero Filipponi, Walther Janous, Bob Prielipp, Sahib Singh, J. Suck, and the proposer.

Diophantine Equation

B-536 Proposed by L. Kuipers, Sierre, Switzerland

Find all solutions in integers \( x \) and \( y \) of

\[ x^4 + 2x^3 + 2x^2 + x + 1 = y^2. \]

Solution by Paul S. Bruckman, Fair Oaks, CA

If we make the substitution

\[ w = 1 + 2x + 2x^2, \]

the given equation is transformed to the simpler one,

\[ 4y^2 - w = 3. \]

Thus, \((2y - w)(2y + w) = 3\), which has only the four solutions:

\[(w, y) = (1, 1), (-1, 1), (-1, -1), (1, -1).\]

Setting \( w = 1 \) in (1) yields \( 2x(x + 1) = 0 \), which implies \( x = 0 \) or \( x = -1 \). This yields four solutions \((x, y)\), given by

\[(x, y) = (0, 1), (0, -1), (-1, 1), (-1, -1).\]

On the other hand, setting \( w = -1 \) in (1) yields \( x^2 + x + 1 = 0 \), which is impossible for integral \( x \) (the solutions being the complex cube roots of unity). Thus, all the integer solutions of the original equation are given by (3).


Another Diophantine Equation

B-537 Proposed by L. Kuipers, Sierre, Switzerland

Find all solutions in integers \( x \) and \( y \) of

\[ x^4 + 3x^3 + 3x^2 + x + 1 = y^2. \]

Solution by John Oman & Bob Prielipp, University of Wisconsin-Oshkosh, WI

We shall show that the only solutions \((x, y)\) in integers of the given equation are \((0, 1), (0, -1), (1, 3), (1, -3), (-1, 1), (-1, -1), (-3, 5), \) and \((-3, -5)\).

It is easily seen that \((x, y)\) is a solution if and only if \((x, -y)\) is a solution. Hence, it suffices to find all solutions \((x, y)\) in integers of the given equation where \( y \geq 0 \).

We begin with the following collection of equivalent equations:
If $x > 1$, $6x^2 + 9x + 2 < 6y$ [by (1)] and $6y < 6x^2 + 9x + 3$ [by (2)]. Hence, there are no solutions when $x > 1$. If $x < -3$, $6x^2 + 9x + 2 < 6y$ [by (1)] and $6y < 6x^2 + 9x + 3$ [by (2)]. Hence, there are no solutions when $x < -3$. The problem is now easily completed.