ON m-TH ORDER LINEAR RECURRENCES

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1. INTRODUCTION

Fix numbers $u_0, u_1, \ldots, u_{-1}$, and for every $n \geqslant 0$, define u_{m+n} by means of the m preceding terms with the rule

$$u_{m+n} - k_1 u_{m+n-1} - \cdots - k_m u_n = 0, \text{ with } k_m \neq 0. \tag{1.1}$$

In this note, we wish to present two formulas for these numbers u_n satisfying the above m-th order linear recurrence (Sections 2 and 3).

These results are probably known to some readers; however, since from time to time we happen to see in the literature special cases of these formulas, it may be worthwhile to present them once and for all.

Note that for $m = 2$, $k_1 = k_2 = 1$, $u_0 = u_1 = 1$, one is dealing with the Fibonacci numbers, which have been extensively studied by many authors (see, for instance, [13], [5], and [3]), and which were used by Matijasevič [9] in his notorious proof that Hilbert's tenth problem is recursively unsolvable.

2. GENERATING FUNCTION AND BINET'S FORMULA

Using the m-th order linear recurrence

$$u_{m+n} = k_1 u_{m+n-1} + k_2 u_{m+n-2} + \cdots + k_m u_n,\ k_m \neq 0, \tag{2.1}$$

(with the k_i's in $\mathbb{Z}$ for instance, or in a given field), we easily obtain

$$\left(\sum_{n=0}^{\infty} u_n X^n\right)(1 - k_1 X - \cdots - k_m X^m) = \sum_{i=0}^{m-1} v_i X^i,$$

where the v_i's, functions of the initial conditions on $u_0, u_1, \ldots, u_{m-1}$, are defined by

$$v_i = -\sum_{j=0}^{i} u_{i-j} k_j, \tag{2.2}$$

(with $k_0 = -1$ throughout this article). Associated with that recursive sequence is the following polynomial,

$$f(X) = X^m = k_1 X^{m-1} - \cdots - k_{m-1} X - k_m = (X - \alpha_1)(X - \alpha_2) \cdots (X - \alpha_m),$$

whose roots we assume distinct (and nonzero, since $k_m \neq 0$).

Then we have

$$\sum_{n=0}^{\infty} u_n X^n = \frac{v_0 + v_1 X + \cdots + v_{m-1} X^{m-1}}{1 - k_1 X - k_2 X^2 - \cdots - k_m X^m}$$

(continued)

$$= \frac{v_0 + v_1X + \cdots + v_{m-1}X^{m-1}}{(1 - \alpha_1X)(1 - \alpha_2X) \cdot \cdots \cdot (1 - \alpha_mX)}$$

$$= (v_0 + v_1X + \cdots + v_{m-1}X^{m-1})(1 + p_1X + \cdots + p_jX^j + \cdots),$$

where p_j stands for the sum of all symmetric functions of weight j in α_1, α_2, ..., α_m; in other words [2], p_j is the "sum of the homogeneous products of j dimensions" of the m symbols α_1, ..., α_m.

Let us recall from Volume 1 of [2, p. 178] that

$$p_j = \sum_{i=1}^{m} \frac{\alpha_i^{m-1+j}}{f'(\alpha_i)} \quad \text{with } p_0 = 1,$$

and that $p_{-1} = p_{-2} = \cdots = p_{-m+1} = 0$ (which follows from Example 4 of p. 172). We therefore obtain for the m-th number u_n what can be called

$$\text{BINET'S FORMULA: } \boxed{u_n = \sum_{j=0}^{m-1} v_j p_{n-j}}.$$

EXAMPLES: (1) Let $v_0 = v_1 = \cdots = v_{m-2} = 0$, $v_{m-1} = 1$; then, as in Formula 9 of [7],

$$u_n = p_{n-m+1} = \frac{\alpha_1^n}{f'(\alpha_1)} + \frac{\alpha_2^n}{f'(\alpha_2)} + \cdots + \frac{\alpha_m^n}{f'(\alpha_m)}.$$

(2) For $m = 2$, $m = 3$, we recover Binet's formulas of [3] and [11].

(3) For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, define s_n by

$$s_n = \alpha_1^n + \alpha_2^n + \cdots + \alpha_m^n.$$

As is well known (see [2]), Newton's formulas state that:

$$s_n = \begin{cases} m & \text{if } n = 0 \\ k_1 & \text{if } n = 1 \\ k_1s_{n-1} + k_2s_{n-2} + \cdots + k_{n-1}s_1 + nk_n & \text{if } 2 \leqslant n \leqslant m - 1 \\ k_1s_{n-1} + k_2s_{n-2} + \cdots + k_ms_{n-m} & \text{if } n \geqslant m. \end{cases}$$

In particular, if $u_n = s_n$, then $\{u_n\}$ satisfies (2.1).

Thus, using the fact that $k_0 = -1$, we find that (2.2) gives:

$$\begin{cases} v_0 = m = -mk_0 \\ v_1 = s_1 - mk_1 = -(m - 1)k_1 \\ v_2 = s_2 - s_1k_1 - mk_2 = -(m - 2)k_2 \\ \vdots \\ v_{m-1} = s_{m-1} - s_{m-2}k_1 - s_{m-3}k_2 - \cdots - s_1k_{m-2} - mk_{m-1} = -1k_{m-1}. \end{cases}$$

In short, for $j = 0, 1, \ldots, m - 1$, $v_j = -(m - j)k_j$, and Binet's formula becomes

$$s_n = -\sum_{j=0}^{m-1}(m - j)k_jp_{n-j}.$$

3. ANOTHER FORMULA

We can also use the multinomial theorem to obtain a formula for u_n that is a function of (i.e., the elementary symmetric functions of) $k_1, k_2, \ldots, k_m$. Here we no longer require that the roots of f be distinct. Within a certain radius of convergence, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} u_n X^n &= \frac{v_0 + v_1 X + \cdots + v_{m-1}X^{m-1}}{1 - (k_1 X + k_2 X^2 + \cdots + k_m X^m)} \\ &= (v_0 + v_1 X + \cdots + v_{m-1}X^{m-1})\left[\sum_{j=0}^{\infty}(k_1 X + \cdots + k_{m-1}X^{m-1} + k_m X^m)^j\right] \\ &= (v_0 + v_1 X + \cdots + v_{m-1}X^{m-1})\left(\sum_{i=0}^{\infty} A(i)X^i\right),\end{aligned}$$

where

$$A(i) = \sum \frac{(t_1 + t_2 + \cdots + t_m)!}{t_1!t_2! \ldots t_m!} k_1^{t_1} k_2^{t_2} \ldots k_m^{t_m},$$

the last sum being taken over all m-tuples $(t_1, t_2, \ldots, t_m)$ of $\mathbb{N}^m$ such that

$$t_1 + 2t_2 + \cdots + mt_m = i.$$

Defining $A(i)$ to be 0 for $i < 0$, we therefore conclude:

$$\boxed{u_n = \sum_{j=0}^{m-1} v_j A(n - j)}. \tag{3.1}$$

EXAMPLES: (1) If $v_0 = 1$, $v_1 = v_2 = \cdots = v_{m-1} = 0$, then

$$u_n = A(n).$$

(2) Let $s_n = \alpha_1^n + \alpha_2^n + \cdots + \alpha_m^n$. Replacing v_j by $-(m-j)k_j$, and making in the j-th summation ($j = 1, \ldots, m-1$) of (3.1) the change of variable $t_j \to t_j - 1$, we obtain after a few calculations what is called in [2] Waring's formula for s_n:

$$s_n = \sum_{t_1 + 2t_2 + \cdots + mt_m = n} \frac{n(t_1 + t_2 + \cdots + t_m - 1)!}{t_1!t_2! \ldots t_m!} k_1^{t_1} k_2^{t_2} \ldots k_m^{t_m}.$$

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