INVERSE TRIGONOMETRICAL SUMMATION FORMULAS INVOLVING PELL POLYNOMIALS

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1. INTRODUCTION

Pell polynomials $P_n(x)$ and *Pell-Lucas Polynomials* $Q_n(x)$ are defined in [3] by the recurrence relation and initial conditions

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x) \qquad P_0(x) = 0,\ P_1(x) = 1 \tag{1.1}$$

and

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x) \qquad Q_0(x) = 2,\ Q_1(x) = 2x. \tag{1.2}$$

Properties of these polynomials are also set out in [3]. Among these, the most important for our current purposes are the following:

$$P_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.3}$$

and *Binet forms*

$$Q_n(x) = \alpha^n + \beta^n \tag{1.4}$$

where

$$\alpha = x + \sqrt{x^2 + 1}, \qquad \beta = x - \sqrt{x^2 + 1} \tag{1.5}$$

are the roots of the characteristic equation

$$\lambda^2 - 2x\lambda - 1 = 0 \tag{1.6}$$

of the recurrences (1.1) and (1.2), so that:

$$\alpha + \beta = 2x, \quad \alpha - \beta = 2\sqrt{x^2 + 1}, \quad \alpha\beta = -1; \tag{1.7}$$

$$P_{n+1}(x)P_{n-1}(x) - P_n^2(x) = (-1)^n, \tag{1.8}$$

and *Simson's formulas*

$$Q_{n+1}(x)Q_{n-1}(x) - Q_n^2(x) = (-1)^{n-1} \cdot 4(x^2 + 1); \tag{1.9}$$

$$P_{n+1}(x) + P_{n-1}(x) = Q_n(x); \tag{1.10}$$

$$Q_{n+1}(x) + Q_{n-1}(x) = 4(x^2 + 1)P_n(x); \tag{1.11}$$

$$P_n(x)Q_n(x) = P_{2n}(x). \tag{1.12}$$

When $x = 1$, $P_n(1) = P_n$ and $Q_n(1) = Q_n$ reduce to the Pell numbers and the "Pell-Lucas" numbers, respectively. On the other hand, $x = \frac{1}{2}$ leads to $P_n(\frac{1}{2}) = F_n$ and $Q_n(\frac{1}{2}) = L_n$, the Fibonacci and Lucas numbers, respectively.

Analogous results to some of those obtained below occur in [2], which provided the stimulus for this article.

2. INVERSE TANGENT AND COTANGENT FORMULAS

Calculation using (1.3) yields

$$P_{2n+1}(x)P_{2n+2}(x) - P_{2n}(x)P_{2n+3}(x) = 2x \qquad (n \geqslant 0). \tag{2.1}$$

Substituting for $P_{2n+3}(x)$ from (1.1) and rearranging, we obtain

$$P_{2n}(x) = \frac{(P_{2n+1}(x)/2x)P_{2n+2}(x) - 1}{(P_{2n+1}(x)/2x) + P_{2n+2}(x)}, \tag{2.2}$$

which can be expressed trigonometrically as

$$\cot^{-1}P_{2n}(x) = \cot^{-1}(P_{2n+1}(x)/2x) + \cot^{-1}P_{2n+2}(x). \tag{2.3}$$

Summing, we derive

$$\sum_{r=0}^{n} \cot^{-1}(P_{2r+1}(x)/2x) = \frac{\pi}{2} - \cot^{-1}P_{2n+2}(x), \tag{2.4}$$

since $\cot^{-1}0 = \pi/2$. Setting $x = 1$ and letting $n \to \infty$, we have a result about Pell numbers P_n:

$$\sum_{r=0}^{\infty} \cot^{-1}(P_{2r+1}/2) = \frac{\pi}{2}, \tag{2.5}$$

while putting $x = \frac{1}{2}$ leads to the known summation formula involving Fibonacci numbers F_n:

$$\sum_{r=0}^{\infty} \cot^{-1}F_{2r+1} = \frac{\pi}{2}. \tag{2.6}$$

Next,

$$\tan^{-1}\left(\frac{1}{P_{2r-2}(x)}\right) - \tan^{-1}\left(\frac{1}{P_{2r}(x)}\right) = \tan^{-1}\left(\frac{P_{2r}(x) - P_{2r-2}(x)}{1 + P_{2r}(x)P_{2r-2}(x)}\right). \tag{2.7}$$

$$= \tan^{-1}\left(\frac{2x}{P_{2r-1}(x)}\right)$$

using (1.1) and (1.8) and simplifying.

Consequently, summation of (2.7) produces

$$\sum_{r=1}^{n} \tan^{-1}(2x/P_{2r-1}(x)) = \frac{\pi}{2} - \tan^{-1}(1/P_{2n}(x)) \tag{2.8}$$

since $P_0(x) = 0$ and $\tan x$ is undefined for $x = \pi/2$.

Alternatively, (2.8) is a direct consequence of (2.4).

As above, the special cases $x = \frac{1}{2}$, $x = 1$ reduce (2.8) to information about the Fibonacci numbers and the Pell numbers, respectively.

In particular, when $x = \frac{1}{2}$, (2.8) leads to the limiting summation

$$\sum_{r=1}^{\infty} \tan^{-1}\left[\frac{1}{F_{2r-1}}\right] = \frac{\pi}{2},$$

which, like (2.6), is a slight variation of the D. H. Lehmer summation result for Fibonacci numbers (given in [2] as Theorem 5).

When $x = 1$ in (2.8), we obtain another form of (2.5) for Pell numbers.

Furthermore, using (1.9) and (1.11), one may obtain

$$\tan^{-1}\left(\frac{1}{Q_{2r-2}(x)}\right) + \tan^{-1}\left(\frac{1}{Q_{2r}(x)}\right) = \tan^{-1}\left(\frac{4(x^2+1)P_{2r-1}(x)}{Q^2_{2r-1}(x) + 4(x^2+1) - 1}\right). \tag{2.7a}$$

Unfortunately, the right-hand side does not simplify any further as we should have desired, by comparison with (2.7). However, if we choose $x = \frac{1}{2}$ [so that $P_r(\frac{1}{2}) = F_r$, $Q_r(\frac{1}{2}) = L_r$], then the equation reduces to

$$\begin{aligned}\tan^{-1}\left(\frac{1}{L_{2r-2}}\right) + \tan^{-1}\left(\frac{1}{L_{2r}}\right) &= \tan^{-1}\left(\frac{1}{F_{2r-1}}\right)\\ &= \tan^{-1}\left(\frac{1}{F_{2r-2}}\right) - \tan^{-1}\left(\frac{1}{F_{2r}}\right) \quad \text{from (2.7)},\end{aligned}$$

both of which are given in [2] (as Theorems 3 and 4), in a slightly varied form.

Proceeding to the limiting summation in the first of these equations (with r replaced by $r + 1$) produces the result for Lucas numbers given in [2] as Theorem 6, namely

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{1}{L_{2r}}\right) = \frac{1}{2}\tan^{-1}2 = \tan^{-1}\left(\frac{\sqrt{5}-1}{2}\right).$$

Furthermore, for Pell-Lucas polynomials,

$$\begin{aligned}\tan^{-1}\left(\frac{Q_{r-1}(x)}{Q_r(x)}\right) - \tan^{-1}\left(\frac{Q_r(x)}{Q_{r+1}(x)}\right) &= \tan^{-1}\left(\frac{Q_{r+1}(x)Q_{r-1}(x) - Q_r^2(x)}{Q_r(x)(Q_{r+1}(x) + Q_{r-1}(x))}\right) \qquad (2.9)\\ &= \tan^{-1}\left(\frac{(-1)^{r-1}}{P_{2r}(x)}\right)\end{aligned}$$

by (1.9), (1.11), and (1.12).

Hence,

$$\begin{aligned}\sum_{r=1}^{n} \tan^{-1}\left(\frac{(-1)^{r-1}}{P_{2r}(x)}\right) &= \tan^{-1}\left(\frac{Q_0(x)}{Q_1(x)}\right) - \tan^{-1}\left(\frac{Q_n(x)}{Q_{n+1}(x)}\right) \qquad (2.10)\\ &= \tan^{-1}\left(\frac{P_n(x)}{P_{n+1}(x)}\right)\end{aligned}$$

by (1.3), (1.4), and (1.5), since $1 + x\alpha = \alpha\sqrt{1+x^2}$ and $1 + x\beta = -\beta\sqrt{1+x^2}$.

By (1.5), $\alpha > 0$ and $\beta < 0$ for all real x. Furthermore, $\alpha > 1$ for $x > 0$ and $0 < \alpha < 1$ for $x < 0$. In addition, $|\beta| < 1$ for $x > 0$ and $|\beta| > 1$ for $x < 0$. From these considerations, (1.3), and (2.10), we have

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{(-1)^{r-1}}{P_{2r}(x)}\right) = \begin{cases}\tan^{-1}(1/\alpha), & \text{for } x > 0\\ \tan^{-1}(1/\beta), & \text{for } x < 0.\end{cases} \tag{2.11}$$

An argument similar to that used in deriving (2.10) shows that

$$\sum_{r=1}^{n} \tan^{-1}\left(\frac{(-1)^r}{P_{2r}(x)}\right) = \tan^{-1}\left(\frac{Q_1(x)}{Q_0(x)}\right) - \tan^{-1}\left(\frac{Q_{n+1}(x)}{Q_n(x)}\right) \tag{2.12}$$

$$= \tan^{-1}\left(\frac{xQ_n(x) - Q_{n+1}(x)}{Q_n(x) + xQ_{n+1}(x)}\right) = -\tan^{-1}\left(\frac{P_n(x)}{P_{n+1}(x)}\right).$$

Letting $n \to \infty$, we have another derivation of (2.11). Special manifestations of (2.9), (2.10), (2.11), and (2.12) are derived when $x = \frac{1}{2}$ and $x = 1$, yielding information about the Fibonacci and Lucas numbers, and the Pell and Pell-Lucas numbers, respectively.

For example, if $x = \frac{1}{2}$, then (2.11) with (1.5) yields the known result

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{(-1)^{r-1}}{F_{2r}}\right\} = \frac{1}{2}\tan^{-1}2 = \tan^{-1}\left(\frac{\sqrt{5}-1}{2}\right),$$

which should be compared with the similar result for Lucas numbers preceding (2.9).

If $x = \frac{1}{2}$ in (2.9), then, with r replaced by $n + 1$, Theorem 3 (first part) of [1] results.

When $x = \frac{1}{2}$ in (2.12), we obtain Theorem 4 of [1].

Following the method used for (2.7), with appeal to (1.8) and (1.10), and then summing, we ascertain that

$$\sum_{r=1}^{\infty} (-1)^{r-1}\tan^{-1}\left(\frac{Q_{2r}(x)}{P_{2r}^2(x)}\right) = \frac{\pi}{4} + (-1)^{n-1}\tan^{-1}\left(\frac{1}{P_{2n+1}(x)}\right). \tag{2.13}$$

Summing to infinity gives

$$\sum_{r=1}^{\infty} (-1)^{r-1}\tan^{-1}\left(\frac{Q_{2r}(x)}{P_{2r}^2(x)}\right) = \frac{\pi}{4}, \text{ provided } P_{2n+1}(x) \to \infty.$$

When $x = \frac{1}{2}$, it follows that, for Fibonacci and Lucas numbers,

$$\sum_{r=1}^{\infty} (-1)^{r-1}\tan^{-1}\left(\frac{L_{2r}}{F_{2r}^2}\right) = \frac{\pi}{4}.$$

When $x = 1$, it follows that, for Pell and Pell-Lucas numbers,

$$\sum_{r=1}^{\infty} (-1)^{r-1}\tan^{-1}\left(\frac{Q_{2r}}{P_{2r}^2}\right) = \frac{\pi}{4}.$$

3. GENERALIZATIONS

Results (2.10) and (2.12) can be generalized as indicated below. Firstly, however, some extensions and generalizations of previous formulas must be established. Using the Binet forms (1.3) and (1.4), we may, with due diligence, demonstrate the validity of the following:

$$P_{(r+1)(2k-1)}(x)P_{(r-1)(2k-1)}(x) - P^2_{r(2k-1)}(x) = (-1)^{r(2k-1)}P^2_{2k-1}(x); \tag{3.1}$$

$$Q_{(r+1)(2k-1)}(x)Q_{(r-1)(2k-1)}(x) - Q^2_{r(2k-1)}(x)$$
$$= (-1)^{(r-1)(2k-1)}4(x^2+1)P^2_{2k-1}(x); \tag{3.2}$$

$$P_{r(2k-1)}(x)\{P_{(r+1)(2k-1)}(x) + P_{(r-1)(2k-1)}(x)\} = P_{2r(2k-1)}(x)P_{2k-1}(x); \tag{3.3}$$

$$Q_{r(2k-1)}(x)\{Q_{(r+1)(2k-1)}(x) + Q_{(r-1)(2k-1)}(x)\}$$
$$= 4(x^2+1)P_{2r(2k-1)}(x)P_{2k-1}(x). \tag{3.4}$$

The odd factor $2k-1$ is necessary to ensure the vanishing of certain terms that arise in the course of the algebraic manipulations. Of course, (3.1) and (3.2) are extensions of the Simson's formulas (1.8) and (1.9), respectively, when $k = 1$ ($P_1(x) = 1$).

Now, consider

$$\tan^{-1}\left(\frac{Q_{(r-1)(2k-1)}(x)}{Q_{r(2k-1)}(x)}\right) - \tan^{-1}\left(\frac{Q_{r(2k-1)}(x)}{Q_{(r+1)(2k-1)}(x)}\right) \tag{3.5}$$

$$= \tan^{-1}\left(\frac{Q_{(r+1)(2k-1)}(x)Q_{(r-1)(2k-1)}(x) - Q^2_{r(2k-1)}(x)}{Q_{r(2k-1)}(x)\{Q_{(r+1)(2k-1)}(x) + Q_{(r-1)(2k-1)}(x)\}}\right)$$

$$= \tan^{-1}\left(\frac{(-1)^{(r-1)(2k-1)}\cdot 4(x^2+1)P^2_{2k-1}(x)}{4(x^2+1)P_{2r(2k-1)}(x)P_{2k-1}(x)}\right) \text{ by (3.2), (3.4)}$$

$$= \tan^{-1}\left(\frac{(-1)^{(r-1)(2k-1)}P_{2k-1}(x)}{P_{2r(2k-1)}(x)}\right)$$

Put $k = 1$ in (3.5) and we obtain (2.9).

If we sum (3.5), as before, we have

$$\sum_{r=1}^{n}\tan^{-1}\left(\frac{(-1)^{(r-1)(2k-1)}P_{2k-1}(x)}{P_{2r(2k-1)}(x)}\right) \tag{3.6}$$

$$= \tan^{-1}\left(\frac{Q_0(x)}{Q_{2k-1}(x)}\right) - \tan^{-1}\left(\frac{Q_{n(2k-1)}(x)}{Q_{(n+1)(2k-1)}(x)}\right).$$

Recourse to (3.1) and (3.3) will likewise reveal that

$$\tan^{-1}\left(\frac{(-1)^{r(2k-1)}P_{2k-1}(x)}{P_{2r(2k-1)}(x)}\right) \tag{3.7}$$

$$= \tan^{-1}\left(\frac{P_{(r-1)(2k-1)}(x)}{P_{r(2k-1)}(x)}\right) - \tan^{-1}\left(\frac{P_{r(2k-1)}(x)}{P_{(r+1)(2k-1)}(x)}\right).$$

Therefore, since $P_0(x) = 0$,

$$\sum_{r=1}^{n}\tan^{-1}\left(\frac{(-1)^{r(2k-1)}P_{2k-1}(x)}{P_{2r(2k-1)}(x)}\right) = -\tan^{-1}\left(\frac{P_{n(2k-1)}(x)}{P_{(n+1)(2k-1)}(x)}\right). \tag{3.8}$$

Putting $k = 1$ in (3.6) and (3.8) leads us back to (2.10) and (2.12), respectively. Some advantage accrues in (3.7) if r is replaced by $r + 1$. Summation then leaves an additional (nonvanishing) term

$$\left[= \tan^{-1}\left(\frac{P_{2k-1}(x)}{P_{2(2k-1)}(x)}\right)\right]$$

on the right-hand side. If we put $k = 1$ in (3.7) and replace r by $r + 1$, then $x = 1$ gives us

$$\tan^{-1}\left(\frac{P_r}{P_{r+1}}\right) - \tan^{-1}\left(\frac{P_{r+1}}{P_{r+2}}\right) = \tan^{-1}\left(\frac{(-1)^{r+1}}{P_{2r+2}}\right), \tag{3.7a}$$

while $x = \frac{1}{2}$ gives Theorem 3 (second part) in [1].

In conclusion, we notice, using the results needed for (2.11), that

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{(-1)^{r(2k-1)}P_{2k-1}(x)}{P_{2r(2k-1)}(x)}\right) = -\begin{cases} \tan^{-1}(1/\alpha^{2k-1}), & \text{for } x > 0 \\ \tan^{-1}(1/\beta^{2k-1}), & \text{for } x < 0. \end{cases} \tag{3.9}$$

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