ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications concerning ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$ 

Also, $a$ and $b$ designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-562 Proposed by Herta T. Freitag, Roanoke, VA

Let $c_0$ be the integer in $\{0, 1, 2, 3, 4\}$ such that

$$c_0 \equiv L_{2n} + [n/2] - [(n - 1)/2] \pmod{5},$$

where $[x]$ is the greatest integer in $x$. Determine $c_n$ as a function of $n$.

B-563 Proposed by Herta T. Freitag, Roanoke, VA

Let $S_n = \sum_{i=1}^{n} L_{2i+1}L_{2i-2}$. For which values of $n$ is $S_n$ exactly divisible by 4?

B-564 Proposed by László Cseh, Cluj, Romania

Let $a = (1 + \sqrt{5})/2$ and $[x]$ be the greatest integer in $x$. Prove that

$$[aF_1] + [aF_2] + \cdots + [aF_n] = F_{n+3} - [(n + 4)/2].$$

B-565 Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany

Let $P_0$, $P_1$, $\ldots$ be the sequence of Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \in \{2, 3, \ldots\}$. Show that
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\[ 9 \sum_{k=0}^{n} P_k F_k = P_{n+2} F_n + P_{n+1} F_{n+2} + P_{n} F_{n-1} - P_{n-1} F_{n+1}. \]

**B-566 Proposed by Heinz-Jürgen Seiffert, Berlin, Germany**

Let \( P_n \) be as in B-565. Show that

\[ 9 \sum_{k=0}^{n} P_k L_k = P_{n+2} L_n + P_{n+1} L_{n+2} + P_n L_{n-1} - P_{n-1} L_{n+1} - 6. \]

**B-567 Proposed by P. Rubio, Dragados Y Construcciones, Madrid, Spain**

Let \( a_n = a_1 = 1 \) and \( a_{n+1} = a_n + na_{n-1} \) for \( n \) in \( \mathbb{Z}^+ = \{1, 2, \ldots\} \). Find a simple formula for

\[ G(x) = \sum_{k=0}^{n} a_k \mathbb{I}^{\alpha k}. \]

**SOLUTIONS**

**Lucas Geometric Progression**

**B-538 Proposed by Herta T. Freitag, Roanoke, VA**

Prove that \( \sqrt{5} g^n = g L_n + L_{n-1} \), where \( g \) is the golden ratio \( (1 + \sqrt{5})/2 \).

**Solution by László Cseh, Cluj, Romania**

It is well known that \( L_n = g^n + \bar{g}^n \), where \( \bar{g} = (1 - \sqrt{5})/2 \). Now

\[ g L_n + L_{n-1} = g^{n+1} + g \cdot \bar{g} \cdot g^{n-1} + g^{n-1} + \bar{g}^{n-1} \]

\[ = g^{n+1} - \bar{g}^{n+1} - g^n + g^n + \bar{g}^{n-1} \]

\[ = g^n(g + g^{-1}) = \sqrt{5} g^n. \] Q.E.D.

**Remark:** By a similar argument, it can be proved that \( g^n = g P_n + P_{n-1} \).


Not Necessarily Golden GP's

**B-539 Proposed by Herta T. Freitag, Roanoke, VA**

Let \( g = (1 + \sqrt{5})/2 \) and show that

\[ \left[ 1 + 2 \sum_{i=1}^{m} g^{-3i} \right] \left[ 1 + 2 \sum_{i=1}^{m} (-1)^i g^{-3i} \right] = 1. \]

**Solution by A. G. Shannon, NSWIT, Sydney, Australia**

\[ \left[ 1 + 2 \sum_{i=1}^{m} g^{-3i} \right] \left[ 1 + 2 \sum_{i=1}^{m} (-1)^i g^{-3i} \right] |g| < 1 \]

(continued)
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\[
\begin{align*}
\left[ 1 + \frac{2}{g^3} \right] \left[ 1 - \frac{2}{g^3 + 1} \right] &= \text{(sums of GPs)} \\
\left[ \frac{g^3 + 1}{g^3 - 1} \right] \left[ \frac{g^3 - 1}{g^3 + 1} \right] &= 1, \text{ as required.}
\end{align*}
\]

This holds for \(|g| < 1\); i.e., \(g\) does not have to equal \(a\).


Product of 3 Successive Integers

B-540 Proposed by A. B. Patel, V.S. Patel College of Arts & Sciences, Bilimora, India

For \(n = 2, 3, \ldots\), prove that

\[
P_{n-1}P_nP_{n+1}L_nL_{n+1}L_{n+2}
\]

is not a perfect square.

Solution by L. A. G. Dresel, University of Reading, England

Using the identities \(F_nL_n = F_{2n}\) and \(F_{2n-2}F_{2n+2} = F_{2n}^2 - 1\), we have

\[
P = P_{n-1}P_nP_{n+1}L_nL_{n+1}L_{n+2} = F_{2n}F_{2n-2}F_{2n+2} = F_{2n}(F_{2n}^2 - 1).
\]

Now for \(n = 2, 3, \ldots\), we have \(F_{2n} > 1\) and, therefore, \((F_{2n}^2 - 1)\) is not a perfect square; furthermore, \(F_{2n}^2 - 1 = (F_{2n} - 1)(F_{2n} + 1)\) is coprime to \(F_{2n}\) and, therefore, the expression \(P\) is not a perfect square.


Congruence Modulo 9

B-541 Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany

Show that \(P_{n+3} + P_{n+1} + P_n \equiv 3(-1)^n L_n (\text{mod } 9)\), where the \(P_n\) are the Pell numbers defined by \(P_0 = 0, P_1 = 1\), and

\[
P_{n+2} = 2P_{n+1} + P_n \text{ for } n \in \mathbb{N} = \{0, 1, 2, \ldots\}.
\]

Solution by L. A. G. Dresel, University of Reading, England

\[
P_{n+3} + P_{n+1} + P_n = 2P_{n+2} + 2P_{n+1} + P_n = 3P_{n+2}.
\]

Let \(K_n = (-1)^n L_n\). Then since \(P_{n+2} = L_{n+1} + L_n\), multiplying by \((-1)^n\) we obtain \(K_{n+2} = -K_{n+1} + K_n\), so that \(K_{n+2} \equiv 2K_{n+1} + K_n (\text{mod } 3)\). Thus, \(K_n\) and \(P_n\) satisfy the same recurrence relation modulo 3, and furthermore,

\[
P_2 = 2P_1 + P_0 = 2 = K_0 \quad \text{and} \quad P_3 = 2P_2 + P_1 = 5 \equiv -1 = K_1 (\text{mod } 3).
\]

It follows that \(P_{n+2} \equiv K_1 (\text{mod } 3)\) for \(n \in \mathbb{N} = \{0, 1, 2, \ldots\}\) and, therefore,
3P\textsubscript{n+2} \equiv 3K\textsubscript{n} (mod 9) for n \in \mathbb{N}, so that

\[ P_{n+3}^3 + P_{n+1}^3 + P_n^3 \equiv 3(-1)^n L_n (mod 9). \]


3rd Order Nonhomogeneous Recursion

B-542 Proposed by Ioan Tomescu, University of Bucharest, Romania

Find the sequence satisfying the recurrence relation

\[ u(n) = 3u(n - 1) - u(n - 2) - 2u(n - 3) + 1 \]

and the initial conditions \( u(0) = u(1) = u(2) = 0. \)

Solution by C. Georgioud, University of Patras, Greece

It is easy to see that the roots of the characteristic polynomial of the homogeneous equation are \( r_1 = 2, r_2 = a, \) and \( r_3 = b \) and that a particular solution of the inhomogeneous equation is \( u_F(n) = 1. \) Therefore, the general solution of the given recurrence relation is

\[ u(n) = A2^n + BF_n + CL_n + 1. \]

The initial conditions give \( A = 1, B = -2, \) and \( C = -1, \) and the solution is

\[ u(n) = 2^n - 2F_n - L_n + 1 = 2^n - F_{n+3} + 1. \]


Fibonacci Exponential Generating Function

B-543 Proposed by P. Rubio, Dragados Y Construcciones, Madrid, Spain

Let \( a_0 = a_1 = 1 \) and \( a_{n+1} = a_n + a_{n-1} \) for \( n \) in \( \mathbb{Z}^+ = \{1, 2, \ldots \}. \) Find a simple formula for

\[ G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k. \]

Solution by Paul S. Bruckman, Fair Oaks, CA

We see readily that \( a_n = F_{n+1}. \) Hence,

\[ G(x) = \sum_{k=0}^{\infty} \frac{F_{k+1}}{k!} x^k = 5^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(a^{k+1} - \beta^{k+1})x^k}{k!} = 5^{-\frac{1}{2}(\alpha e^{ax} - \beta e^{bx})}. \]