ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications concerning ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-562 *Proposed by Herta T. Freitag, Roanoke, VA*

Let c_n be the integer in $\{0, 1, 2, 3, 4\}$ such that

$$c_n \equiv L_{2n} + [n/2] - [(n-1)/2] \pmod 5,$$

where $[x]$ is the greatest integer in x. Determine c_n as a function of n.

B-563 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_n = \sum_{i=1}^{n} L_{2i+1}L_{2i-2}$. For which values of n is S_n exactly divisible by 4?

B-564 *Proposed by Lászlő Cseh, Cluj, Romania*

Let $a = (1 + \sqrt{5})/2$ and $[x]$ be the greatest integer in x. Prove that

$$[aF_1] + [aF_2] + \cdots + [aF_n] = F_{n+3} - [(n+4)/2].$$

B-565 *Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany*

Let $P_0, P_1, \ldots$ be the sequence of Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \in \{2, 3, \ldots\}$. Show that

$$9\sum_{k=0}^{n} P_k F_k = P_{n+2}F_n + P_{n+1}F_{n+2} + P_n F_{n-1} - P_{n-1}F_{n+1}.$$

B-566 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany*

Let P_n be as in B-565. Show that

$$9\sum_{k=0}^{n} P_k L_k = P_{n+2}L_n + P_{n+1}L_{n+2} + P_n L_{n-1} - P_{n-1}L_{n+1} - 6.$$

B-567 *Proposed by P. Rubio, Dragados Y Construcciones, Madrid, Spain*

Let $a_0 = a_1 = 1$ and $a_{n+1} = a_n + na_{n-1}$ for n in $Z^+ = \{1, 2, \ldots\}$. Find a simple formula for

$$G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!}x^k.$$

SOLUTIONS

Lucas Geometric Progression

B-538 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that $\sqrt{5}g^n = gL_n + L_{n-1}$, where g is the golden ratio $(1 + \sqrt{5})/2$.

Solution by László Cseh, Cluj, Romania

It is well known that $L_n = g^n + \bar{g}^n$, where $\bar{g} = (1 - \sqrt{5})/2$. Now

$$\begin{aligned} gL_n + L_{n-1} &= g^{n+1} + g \cdot \bar{g} \cdot \bar{g}^{n-1} + g^{n-1} + \bar{g}^{n-1} \\ &= g^{n+1} - \bar{g}^{n-1} + g^{n-1} + \bar{g}^{n-1} \\ &= g^n(g + g^{-1}) = \sqrt{5}g^n. \quad \text{Q.E.D.} \end{aligned}$$

Remark: By a similar argument, it can be proved that $g^n = gF_n + F_{n-1}$.

Also solved by Wray G. Brady, Paul S. Bruckman, L. A. G. Dresel, Russell Euler, Piero Filipponi, C. Georghiou, Walther Janous, Hans Kappus, L. Kuipers, Graham Lord, I. Merenyi, George N. Philippou, Bob Prielipp, Heinz-Jürgen Seiffert, A. G. Shannon, Lawrence Somer, W. R. Utz, and the proposer.

Not Necessarily Golden GP's

B-539 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $g = (1 + \sqrt{5})/2$ and show that

$$\left[1 + 2\sum_{i=1}^{\infty} g^{-3i}\right]\left[1 + 2\sum_{i=1}^{\infty}(-1)^i g^{-3i}\right] = 1.$$

Solution by A. G. Shannon, NSWIT, Sydney, Australia

$$\left[1 + 2\sum_{i=1}^{\infty} g^{-3i}\right]\left[1 + 2\sum_{i=1}^{\infty}(-1)^i g^{-3i}\right] \qquad |g| < 1$$

(continued)

$$= \left[1 + \frac{2}{g^3 - 1}\right]\left[1 - \frac{2}{g^3 + 1}\right] \quad \text{(sums of GPs)}$$

$$= \left[\frac{g^3 + 1}{g^3 - 1}\right]\left[\frac{g^3 - 1}{g^3 + 1}\right] = 1, \text{ as required.}$$

This holds for $|g| < 1$; i.e., g does not have to equal α.

Also solved by Wray G. Brady, Paul S. Bruckman, Lászlő Cseh, L. A. G. Dresel, Russell Euler, Piero Filipponi, C. Georghiou, Walther Janous, L. Kuipers, Graham Lord, I. Merenyi, George N. Philippou, Bob Prielipp, Heinz-Jürgen Seiffert, Lawrence Somer, and the proposer.

Product of 3 Successive Integers

B-540 *Proposed by A. B. Patel, V.S. Patel College of Arts & Sciences, Bilimora, India*

For $n = 2, 3, \ldots$, prove that

$$F_{n-1}F_nF_{n+1}L_{n-1}L_nL_{n+1}$$

is not a perfect square.

Solution by L. A. G. Dresel, University of Reading, England

Using the identities $F_nL_n = F_{2n}$ and $F_{2n-2}F_{2n+2} = F_{2n}^2 - 1$, we have

$$P = F_{n-1}F_nF_{n+1}L_{n-1}L_nL_{n+1} = F_{2n-2}F_{2n}F_{2n+2} = F_{2n}(F_{2n}^2 - 1).$$

Now for $n = 2, 3, \ldots$, we have $F_{2n} > 1$ and, therefore, $(F_{2n}^2 - 1)$ is not a perfect square; furthermore, $F_{2n}^2 - 1 = (F_{2n} - 1)(F_{2n} + 1)$ is coprime to F_{2n} and, therefore, the expression P is not a perfect square.

Also solved by Wray G. Brady, Paul S. Bruckman, Adina Di Porto & Piero Filipponi, Walther Janous, L. Kuipers, Bob Prielipp, A. G. Shannon, Lawrence Somer, and the proposer.

Congruence Modulo 9

B-541 *Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany*

Show that $P_{n+3} + P_{n+1} + P_n \equiv 3(-1)^n L_n \pmod 9$, where the P_n are the Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and

$$P_{n+2} = 2P_{n+1} + P_n \text{ for } n \text{ in } N = \{0, 1, 2, \ldots\}.$$

Solution by L. A. G. Dresel, University of Reading, England

$$P_{n+3} + P_{n+1} + P_n = 2P_{n+2} + 2P_{n+1} + P_n = 3P_{n+2}.$$

Let $K_n = (-1)^n L_n$. Then since $L_{n+2} = L_{n+1} + L_n$, multiplying by $(-1)^n$ we obtain $K_{n+2} = -K_{n+1} + K_n$, so that $K_{n+2} \equiv 2K_{n+1} + K_n \pmod 3$. Thus, K_n and P_n satisfy the same recurrence relation modulo 3, and furthermore,

$$P_2 = 2P_1 + P_0 = 2 = K_0 \text{ and } P_3 = 2P_2 + P_1 = 5 \equiv -1 = K_1 \pmod 3.$$

It follows that $P_{n+2} \equiv K \pmod 3$ for n in $N = \{0, 1, 2, \ldots\}$ and, therefore,

$3P_{n+2} \equiv 3K_n \pmod 9$ for n in N, so that

$$P_{n+3} + P_{n+1} + P_n \equiv 3(-1)^n L_n \pmod 9.$$

Also solved by László Cseh, Herta T. Freitag, C. Georghiou, Walther Janous, L. Kuipers, Imre Merenyi, George N. Philippou, Bob Prielipp, A. G. Shannon, Lawrence Somer, and the proposer.

3rd Order Nonhomogeneous Recursion

B-542 *Proposed by Ioan Tomescu, University of Bucharest, Romania*

Find the sequence satisfying the recurrence relation

$$u(n) = 3u(n-1) - u(n-2) - 2u(n-3) + 1$$

and the initial conditions $u(0) = u(1) = u(2) = 0$.

Solution by C. Georghiou, University of Patras, Greece

It is easy to see that the roots of the characteristic polynomial of the homogeneous equation are $r_1 = 2$, $r_2 = a$, and $r_3 = b$ and that a particular solution of the inhomogeneous equation is $u_p(n) = 1$. Therefore, the general solution of the given recurrence relation is

$$u(n) = A2^n + BF_n + CL_n + 1.$$

The initial conditions give $A = 1$, $B = -2$, and $C = -1$, and the solution is

$$u(n) = 2^n - 2F_n - L_n + 1 = 2^n - F_{n+3} + 1.$$

Also solved by Wray G. Brady, Paul S. Bruckman, Odoardo Brugia & Piero Filipponi, László Cseh, L. A. G. Dresel, Russell Euler, Walther Janous, Hans Kappus, L. Kuipers & Peter J. S. Shiue, I. Merenyi, Bob Prielipp, Heinz-Jürgen Seiffert, A. G. Shannon, and the proposer.

Fibonacci Exponential Generating Function

B-543 *Proposed by P. Rubio, Dragados Y Construcciones, Madrid, Spain*

Let $\alpha_0 = \alpha_1 = 1$ and $\alpha_{n+1} = a_n + \alpha_{n-1}$ for n in $Z^+ = \{1, 2, \ldots\}$. Find a simple formula for

$$G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k.$$

Solution by Paul S. Bruckman, Fair Oaks, CA

We see readily that $a_n = F_{n+1}$. Hence,

$$G(x) = \sum_{k=0}^{\infty} F_{k+1} \frac{x^k}{k!} = 5^{-1/2} \sum_{k=0}^{\infty} (\alpha^{k+1} - \beta^{k+1}) \frac{x^k}{k!} = 5^{-1/2} (\alpha e^{\alpha x} - \beta e^{\beta x}).$$

Also solved by Wray G. Brady, O. Brugia & A. Di Porto & P. Filipponi, John R. Burke, László Cseh, L. A. G. Dresel, Russell Euler, C. Georghiou, Walther Janous, Hans Kappus, L. Kuipers, Graham Lord, Imre Merenyi, A. G. Shannon, and the proposer.

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