ON TWO- AND FOUR-PART PARTITIONS OF NUMBERS EACH PART A SQUARE

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1. INTRODUCTION

For each given pair of positive integers k, n, with $k \leqslant n$, a k-part partition of n is a k-element multi-set of positive integers whose sum is n; e.g., all of the 3-part partitions of 7 are: [5, 1, 1], [4, 2, 1], [3, 3, 1], and [3, 2, 2]. In this paper we are especially interested in k-part partitions of numbers for which $k = 2, 4$ and all of the parts are squares. We briefly refer to these as 2-square and 4-square partitions of a number. Thus, [4, 1] is a 2-square partition of 5. Also, recall that for each positive integer n, $\sigma(n)$ denotes the sum of all positive divisors of n.

We are now prepared to state our results.

Theorem 1: A nonsquare odd number n has an odd number of 2-square partitions if and only if $\sigma(n)$ is twice an odd number, i.e., $n = p^e m^2$, $e, m, p \in \mathbb{Z}^+$, p a prime, $p \nmid m$, and $p \equiv e \equiv 1 \pmod 4$.

Theorem 2: If a is odd and not of the form $j(3j \pm 2)$, then $3a + 1$ has an odd number of 4-square partitions of the form

$$3a + 1 = 3j^2 + (6k \pm 1)^2, \quad j, k \in \mathbb{Z}^+$$

if and only if a is a square.

In Section 2, we prove these theorems, and also deduce Fermat's classical two-square theorem as an immediate corollary of Theorem 1.

2. PROOFS OF THEOREMS 1 AND 2

Our proofs are based on two recurrences for the sum-of-divisors function. These recurrences are best stated with the aid of several auxiliary arithmetical functions, which we now define.

Definition: For each positive integer n, $b(n)$ denotes the exponent of the highest power of 2 dividing n; and, $O(n)$ is then defined by the equation

$$n = 2^{b(n)} O(n).$$

Hence, $b(n)$ is a nonnegative integer and $O(n)$ is odd. We now define the arithmetical functions ω and ρ by:

$$\omega(n) = \sigma(n) + \sigma(O(n)), \quad \rho(n) = 3\sigma(n) - 5\sigma(O(n)).$$

The two recurrences are, for each positive integer m:

$$(1)\quad \sigma(2m-1) - \sum_{k=1}\omega(2m-1-(2k-1)^2) + 2\sum_{k=1}\sigma(2m-1-(2k)^2) = \begin{cases} j^2, & \text{if } 2m-1=j^2, \\ 0, & \text{otherwise.}\end{cases}$$

$$(2)\quad \sigma(2m-1) + \sum_{k=1}(6k+1)\sigma(2m-1-2k(6k+2))$$
$$- \sum_{k=1}(6k-1)\sigma(2m-1-2k(6k-2))$$
$$+ \sum_{k=1}(3k-1)\rho(2m-1-(2k-1)(6k-1))$$
$$- \sum_{k=1}(3k-2)\rho(2m-1-(2k-1)(6k-5))$$
$$= \begin{cases} -j(3j+1)(3j+2)/2, & \text{if } 2m-1=j(3j+2), \\ j(3j-2)(3\ \ -1)/2, & \text{if } 2m-1=j(3j-2), \\ 0, & \text{otherwise.}\end{cases}$$

In both (1) and (2), the sums indexed by k extend over all values of k which cause the arguments of σ, ω, and ρ to be positive. For a proof of (1), see [1, pp. 215-217]. (2) is proved in [2, pp. 679-682], where $\rho(n) = \omega(3, -5; n)$.

Proof of Theorem 1: Assume that $2m + 1$, with $m \geqslant 0$, is nonsquare. Recurrence (1) then becomes

$$(3)\quad \sigma(2m+1) - \sum_{1}\omega(2m+1-(2k-1)^2) + 2\sum_{1}\sigma(2m+1-(2k)^2) = 0.$$

If $\sigma(2m + 1)$ is twice an odd number, say $\sigma(2m + 1) = 4a + 2$, for some $a \geqslant 0$, then (3) becomes

$$2a + 1 - \sum_{1}\frac{\omega(2m+1)-(2k-1)^2)}{2} + \sum_{1}\sigma(2m+1-(2k)^2) = 0.$$

Next, owing to the multiplicativity of σ, $\omega(n) = 2^{b(n)+1}\sigma(O(n))$. Hence, for n even, 4 divides $\omega(n)$. It follows that the sum $\Sigma\sigma(2m+1-(2k)^2)$ is odd and, therefore, contains an odd number of odd summands. But, from the well-known fact: $\sigma(n)$ is odd $\Longleftrightarrow$ n is a square or twice a square, it then follows that there is an odd number of pairs $2k$, $2j - 1$ ($j, k \in \mathbb{Z}^+$) such that

$$2m + 1 = (2k)^2 + (2j - 1)^2.$$

In a word, $2m + 1$ has an odd number of 2-square partitions.

Conversely, if $2m + 1$ has an odd number of 2-square partitions, then recurrence (3) allows us to reverse the steps of the foregoing argument, whence $\sigma(2m + 1) \equiv 2 \pmod 4$; i.e., $\sigma(2m + 1)$ is twice an odd number.

Corollary (Fermat): Each rational prime p of the form $4m + 1$ is expressible as a sum of two squares.

Proof: For such a prime p, $\sigma(p) = p + 1 = 4m + 2 = 2(2m + 1)$. Hence, p has at least one 2-square partition.

Proof of Theorem 2: Assume $2m + 1$, with $m \geqslant 0$, is not of the form $j(3j \pm 2)$. Recurrence (2) then becomes

$$
\begin{aligned}
(4)\quad \sigma(2m+1) &+ \sum_{k=1} (6k+1)\sigma(2m+1-2k(6k+2)) \\
&- \sum_{k=1} (6k-1)\sigma(2m+1-2k(6k-2)) \\
&+ \sum_{k=1} (3k-1)\rho(2m+1-(2k-1)(6k-1)) \\
&- \sum_{k=1} (3k-2)\rho(2m+1-(2k-1)(6k-5)) = 0.
\end{aligned}
$$

If $2m + 1$ is a square, then $\sigma(2m + 1)$ is odd. Now,

$$\rho(n) = 2(3 \cdot 2^{b(n)} - 4)\sigma(O(n)).$$

Hence, the sum

$$\sum_1 (6k+1)\sigma(2m+1-2k(6k+2)) - \sum_1 (6k-1)\sigma(2m+1-2k(6k-2))$$

is odd and therefore contains an odd number of odd summands. In a word, there exists an odd number of pairs j, $k \in \mathbb{Z}^+$ such that

$$2m + 1 = j^2 + 2k(6k \pm 2),$$

or equivalently,

$$3(2m + 1) + 1 = 3j^2 + (6k \pm 1)^2.$$

Conversely, if $3(2m + 1) + 1$ has an odd number of 4-square partitions of the prescribed form, then recurrence (4) allows us to reverse the steps of the foregoing argument. And, then, $2m + 1$ must be a square.

REFERENCES

1. J. A. Ewell. "Recurrences for the Sum of Divisors." *Proc. Amer. Math. Soc.* 64 (1977).
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