ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-397 *Proposed by Paul S. Bruckman, Fair Oaks, CA*

For any positive integer n, define the function F_n on $\mathbb{C}$ as follows:

$$F_n(x) \equiv (g^n - 1)(x), \tag{1}$$

where g is the operator

$$g(x) \equiv x^2 - 2. \tag{2}$$

(Thus, $F_3(x) = \{(x^2 - 2)\}^2 - 2\}^2 - 2 - x = x^8 - 8x^6 + 20x^4 - 16x^2 - x + 2$). Find all 2^n zeros of F_n.

H-398 *Proposed by Ambati Jaya Krishna, Freshman, Johns Hopkins University*

Let

$$a + b + c + d + e = \left(\sum_1^\infty \left(\frac{(-1)^{n+1}}{2n - 1}\frac{2}{3}\cdot 9^{1-n} + 7^{1-2n}\right)\right)^2$$

and

$$a^2 + b^2 + c^2 + d^2 + e^2 = \frac{45}{512}\sum_1^\infty n^{-4},$$

$a, b, c, d, e \in \mathbb{R}$. What are the values of a, b, c, d, and e if e is to attain its maximum value?

H-399 *Proposed by M. Wachtel, Zürich, Switzerland*

The twin sequences: $\dfrac{L_{1+6n} - 1}{2} = 0, 14, 260, 4674, 83880, \ldots$

and $\dfrac{L_{5+6n} - 1}{2} = 5, 99, 1785, 32039, \ldots$

are representable by infinitely many identities, partitioned into several groups of similar structure:

	$\dfrac{L_{1+6n} - 1}{2}$ = identical to:	$\dfrac{L_{5+6n} - 1}{2}$ = identical to:
	<u>Group I</u>	
S_1	$3L_{-3+6n} + \dfrac{5F_{-5+6n} - 1}{2}$	$3L_{1+6n} + \dfrac{5F_{-1+6n} - 1}{2}$
S_2	$61L_{-9+6n} + \dfrac{11L_{-14+6n} - 1}{2}$	$61L_{-5+6n} + \dfrac{11L_{-10+6n} - 1}{2}$
S_3	$1103L_{-15+6n} + \dfrac{105F_{-23+6n} - 1}{2}$	$1103L_{-11+6n} + \dfrac{105F_{-19+6n} - 1}{2}$
S_4	$19801L_{-21+6n} + \dfrac{199L_{-32+6n} - 1}{2}$	$19801L_{-17+6n} + \dfrac{199L_{-28+6n} - 1}{2}$
S_n	...	...

<u>Groups II and III</u> (in addition, there are more groups):

II.	S_1	$L_{-1+6n} + \dfrac{L_{-2+6n} - 1}{2}$	$L_{3+6n} + \dfrac{L_{2+6n} - 1}{2}$
	S_2	$23L_{-7+6n} + \dfrac{15F_{-11+6n} - 1}{2}$	$23L_{-3+6n} + \dfrac{15F_{-7+6n} - 1}{2}$
	S_n	...	...
III.	S_1	$2L_{-1+6n} - \dfrac{5F_{-2+6n} + 1}{2}$	$2L_{3+6n} - \dfrac{5F_{2+6n} + 1}{2}$
	S_2	$24L_{-7+6n} - \dfrac{7L_{-11+6n} + 1}{2}$	$24L_{-3+6n} - \dfrac{7L_{-7+6n} + 1}{2}$
	S_n	...	...

Find the construction rules for S_n for each group.

SOLUTIONS

Sum Formula!

H-373 *Proposed by Andreas N. Philippou, University of Patras, Greece (Vol. 22, no. 3, August 1984)*

For any fixed integers $k \geqslant 0$ and $r \geqslant 2$, set

$$f_{n+1,r}^{(k)} = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + n_k = n}} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \ldots, n_k, r - 1}, \quad n \geqslant 0.$$

Show that

$$f_{n+1,r}^{(k)} = \sum_{\ell=0}^{n} f_{\ell+1,1}^{(k)} f_{n+1-\ell,r-1}^{(k)}, \quad n \geqslant 0.$$

Solution by C. Georghiou, University of Patras, Greece

Note that the definition of $f_{n+\ell,r}^{(k)}$ can be extended to include every positive real number r. Define also

$$f_{n+1}^{(k)} = \delta_{n,0}, \qquad n \geqslant 0,$$

where $\delta_{n,m}$ is the Kronecker symbol. Then we show that

$$f_{n+1,r}^{(k)} = \sum_{\ell=0}^{n} f_{\ell+1,s}^{(k)} f_{n+\ell-1,r-s}^{(k)}, \qquad n \geqslant 0, \tag{*}$$

for any fixed positive integer k and any fixed nonnegative real number r.

We use generating functions. For fixed k and r, let $F_{k,r}(x)$ be the generating function of the sequence $\{f_{n+1,r}^{(k)}\}_{n=0}^{\infty}$. Then

$$F_{k,r}(x) = (1 - x - x^2 - \cdots - x^k)^{-r}.$$

Indeed, for some neighborhood of $x = 0$, we have

$$(1 - x - x^2 - \cdots - x^k)^{-r} = \sum_{n=0}^{\infty} \binom{-r}{n} (-1)^n (x + x^2 + \cdots + x^k)^n$$

$$= \sum_{n=0}^{\infty} \binom{r+n-1}{n} \sum_{n_1+n_2+\cdots+n_k=n} \binom{n_1 + n_2 + \cdots + n_k}{n_1, n_2, \ldots, n_k} x^{n_1+2n_2+\cdots+kn_k}$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \binom{n_1 + n_2 + \cdots + n_k + r - 1}{n_1 + n_2 + \cdots + n_k}$$

$$\times \binom{n_1 + n_2 + \cdots + n_k}{n_1, n_2, \ldots, n_k} x^{n_1+2n_2+\cdots+kn_k}$$

$$= \sum_{n=0}^{\infty} x^n \sum_{n_1+2n_2+\cdots+kn_k=n} \binom{n_1 + n_2 + \cdots + n_k + r - 1}{n_1, n_2, \ldots, n_k, r - 1}.$$

Equation (*) follows from

$$F_{k,r}(x) = F_{k,s}(x)F_{k,r-s}(x).$$

Note also that the restriction for $r \geqslant 0$ can be relaxed to r any real number.

Also solved by P. Bruckman.

Bounds of Joy

H-374 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC (Vol. 22, no. 3, August 1984)*

If $\sigma^*(n)$ is the sum of the unitary divisors of n, then

$$\sigma^*(n) = \prod_{p^e \| n} (1 + p^e),$$

where p^e is the highest power of the prime p that divides n. The ratio $\sigma^*(n)/n$ increases as new primes are introduced as factors of n, but decreases as old prime factors appear more often. As N increases, is $\sigma^*(N!)/N!$ bounded or unbounded?

Solution by the proposer.

The primes between $N/2$ and N divide $N!$ exactly once, and those not exceeding $N/2$ divide $N!$ more than once. By considering special cases for N (mod 4), it is easy to show by telescoping products that

$$\prod_{N/2<p\leqslant N}(p+1)/p < \prod_{N/2<2k+1\leqslant N}(2k+2)/(2k+1) < \sqrt{\frac{N+2}{[N/2]+1}} < 1.5$$

if $N \geqslant 6$. Also

$$\prod_{\substack{p^e \| n \\ p\leqslant N/2}} (1+p^{-e}) < \prod_{p \text{ prime}} (1+p^{-2}) = \prod_p (1-p^{-4})/(1-p^{-2})$$
$$= \zeta(2)/\zeta(4) = 15/\pi^2 < 1.52.$$

Therefore,

$$\sigma^*(N!)/N! < (1.52)(1.5) = 2.28$$

if $N \geqslant 6$. The cases $1 \leqslant N \leqslant 5$ are easily checked, so $\sigma^*(N!)/N! < 2.28$ for all N. (Actually, the best bound is 2, achieved for $N = 3$.)

Also solved by P. Bruckman who remarked that $\sigma(N!)/N!$ is unbounded.

Conjectures No More

H-375 *Proposed by Piero Filipponi, Rome, Italy (Vol. 22, no. 3, August 1984)*

Conjecture 1

If $F_k \equiv 0 \pmod{k}$ and $k \neq 5^n$, then $k \equiv 0 \pmod{12}$.

Conjecture 2

Let $m > 1$ be odd. Then, $F_{12m} \equiv 0 \pmod{12m}$ implies either 3 divides m or 5 divides m.

Conjecture 3

Let $p > 5$ be a prime such that $p \nmid F_{24}$, then $F_{12m} \not\equiv 0 \pmod{12m}$.

Conjecture 4

If $L_k \equiv 0 \pmod{k}$, then $k \equiv 0 \pmod{6}$ for $k > 1$.

Solution by Lawrence Somer, Washington, D.C.

In answering the conjectures, we will make use of several definitions and known results. The rank of apparition of k in $\{F_n\}$, denoted by $a(k)$, is the least positive integer m such that $k|F_m$. The prime p is a primitive divisor of F_n if $p|F_n$, but $p \nmid F_m$ for $1 \leqslant m < n$. The following theorem will be the main result we will use and is given by D. Jarden as Theorem A in his paper "Divisibility of Fibonacci and Lucas Numbers by Their Subscripts" [2, pp. 68-75].

Theorem 1: Let $p_1, p_2, \ldots, p_n$ be the distinct primes dividing k, where $k > 1$. Then $k|F_k$ if and only if

$$[a(p_1), a(p_2), \ldots, a(p_n)]|k,$$

where $[a, b, \ldots]$ denotes the least common multiple of $a, b, \ldots$.

We will also need the following propositions.

Proposition 1: Let $m \geqslant 3$. Then $F_m|F_n$ if and only if $m|n$.

Proposition 2: Let

$$k = \prod_{i=1}^{m} p_i^{n_i}$$

be the canonical factorization of k into prime powers. Let r_i be the highest power of p_i dividing $F_{a(p_i)}$ for $1 \leqslant i \leqslant m$. Then

$$a(k) = \underset{1 \leqslant i \leqslant m}{\mathrm{LCM}} \left\{a(p_i)p_i^{\max(0,\, n_i - r_i)}\right\}.$$

Proposition 3: If p is a prime and $p \neq 2$ or 5, then the prime factors of $a(p)$ are less than p.

Proposition 4: If $n \neq 1, 2, 6$, or 12, then F_n has a primitive prime divisor.

Proposition 1 is well known. Propositions 2 and 3 are given by Jarden in [2, p. 68]. Proposition 4 is proved by Carmichael [1, p. 61].

Conjecture 1: FALSE. There is an infinite number of counterexamples. By Theorem F in Jarden's paper [2, p. 72], if $k|F_k$, then $12|k$ or $5|k$. Thus, in any counterexample to Conjecture 1, 5 must divide k. Let $n \geqslant 2$. Let the divisors of 5^n that are unequal to 5 be denoted by $p_1, p_2, \ldots, p_m$. Since $F_5 = 5$, such prime divisors exist by Proposition 4. Let

$$k = 5^{r_n} \sum_{i=1}^{m} p_i^{t_i}, \tag{1}$$

where $r_n \geqslant n$ and at least one of the t_i's $\geqslant 1$. It follows from Theorem 1 and Propositions 1, 2, and 4 that k is a counterexample to Conjecture 1. Clearly, there is an infinite number of such counterexamples. In particular, by a table of the factorizations of Fibonacci numbers given by Jarden [2, pp. 36-59], the only primitive prime divisor of F_{25} is 3001 and the only primitive prime divisor of F_{125} is 158414167964045700001. Then, by (1),

$$k_1 = 5^{r_1} 3001^{s_1} \tag{2}$$

and

$$k_2 = 5^{r_2} 3001^{s_2} 158414167964045700001^{t_2} \tag{3}$$

are each counterexamples to Conjecture 1, where

$$r_1 \geqslant 2,\ s_1 \geqslant 1,\ r_2 \geqslant 3,\ s_2 \geqslant 0, \text{ and } t_2 \geqslant 1.$$

We now provide another infinite class of counterexamples to Conjecture 1. Suppose that k is a counterexample to Conjecture 1. Let $q_1, q_2, \ldots, q_d$ be distinct primes such that $q_i \nmid k$ and q_i is a primitive divisor of F_{k_i}, where $1 \leqslant i \leqslant d$ and $k_i | k$. By Proposition 4, such q_i's exist. Then, by Theorem 1,

$$k' = k \prod_{i=1}^{d} q_i^{n_i} \tag{4}$$

is also a counterexample to Conjecture 1, where at least one of the n_i's $\geqslant 1$. One can show that all counterexamples to Conjecture 1 are of the forms given in (1) or (4). Since $k|F_k$, it follows by (4) and Propositions 1 and 4 that F_k is also a counterexample to Conjecture 1. Let $F(n)$ denote F_n, $F(F(n)) = F^{(2)}(n)$ denote F_{F_n} and so on. Then by (2), (3), and Proposition 4,

$$F^{(r)}(5^{r_1} 3001^{s_1}) \tag{5}$$

and

$$F^{(s)}(5^{r_2} 3001^{s_2} 158414167964045700001^{t_2}) \tag{6}$$

are each explicit counterexamples to Conjecture 1, where

$$r_1 \geqslant 2,\ r_2 \geqslant 2,\ s \geqslant 1,\ s_2 \geqslant 0,\ t_2 \geqslant 1,$$

and either it is the case that $r \geqslant 2$ and $s_1 \geqslant 0$ or it is the case that $s_1 \geqslant 1$ and $r \geqslant 1$.

Conjecture 2: TRUE. Suppose that Conjecture 2 were false. Then $m > 1$ and all the prime factors of m are greater than 5. Let p be the smallest prime factor of m. By Theorem 1, $\alpha(p)|12m$. By Proposition 3, each prime factor of $\alpha(p)$ is less than p. It thus follows that $\alpha(p)$ is relatively prime to m and hence, $\alpha(p)|12$. However, $F_1 = F_2 = 1$ and the only prime divisors of F_3, F_4, F_6, or F_{12} are 2 or 3. We thus have a contradiction and the result follows.

Conjecture 3: This does not make sense as stated.

Conjecture 4: TRUE, by Theorem F in Jarden's paper [2, p. 72]. Theorem F further states that if $L_k \equiv 0 \pmod{k}$, then $4 \nmid k$.

References

1. R. D. Carmichael. "On the Numerical Factors of the Arithmetic Forms $\alpha^n \pm \beta^n$." *Ann. Math.* Second Series *15* (1913):30-70.
2. D. Jarden. *Recurring Sequences.* 3rd ed. Jerusalem: Riveon Lematematika, 1973

Also solved by P. Bruckman and L. Dresel.

New Construction

H-376 *Proposed by H. Klauser, Zürich, Switzerland*
(Vol. 22, no. 4, November 1984)

Let (a, b, c, d) be a quadruple of integers with the property that

$$(a^3 + b^3 + c^3 + d^3) = 0.$$

Clearly, at least one integer must be negative.

Examples: (3, 4, 5, 6), (9, 10, -1, -12)

Find a construction rule so that:

1. out of two given quadruples a new quadruple arises;
2. out of the given quadruple a new quadruple arises.

Solution by Paul Bruckman, Fair Oaks, CA

We let S denote the set of all quadruples $(a, b, c, d) \in \mathbb{Z}^4$ such that

$$a^3 + b^3 + c^3 + d^3 = 0. \tag{1}$$

Lemma 1: Given $(a, b, c, d) \in S$, $(a', b', c', d') \in S$, let

$$\begin{aligned} p &= a(a')^2 + b(b')^2 + c(c')^2 + d(d')^2, \\ q &= a^2a' + b^2b' + c^2c' + d^2d'. \end{aligned} \tag{2}$$

Also, let

$$a'' = pa - qa', \; b'' = pb - qb', \; c'' = pc - qc', \; d'' = pd - qd'. \tag{3}$$

Then

$$(a'', b'', c'', d'') \in S. \tag{4}$$

Proof:
$$\begin{aligned} &(a'')^3 + (b'')^3 + (c'')^3 + (d'')^3 \\ &= p^3(a^3 + b^3 + c^3 + d^3) - q^3\{(a')^3 + (b')^3 + (c')^3 + (d')^3\} \\ &\quad - 3p^2q(a^2a' + b^2b' + c^2c' + d^2d') \\ &\quad + 3pq^2\{a(a')^2 + b(b')^2 + c(c')^2 + d(d')^2\} \\ &= p^3 \cdot 0 - q^3 \cdot 0 - 3p^2q \cdot q + 3pq^2 \cdot p = 0. \end{aligned}$$

This shows that $(a'', b'', c'', d'') \in S$ given by (2) and (3) may be constructed from the given quadruples $(a, b, c, d) \in S$ and $(a', b', c', d') \in S$, solving

Part 1 of the problem.

Example: If $(a, b, c, d) = (3, 4, 5, -6)$, $(a', b', c', d') = (-1, 9, 10, -12)$, then $p = -37$, $q = -47$, $(a'', b'', c'', d'') = (-158, 275, 285, -342)$.

Lemma 2: Given $(a, b, c, d) \in S$, let

$$r = ab^2 + bc^2 + cd^2 + da^2, \quad s = a^2b + b^2c + c^2d + d^2a. \tag{5}$$

Also, let

$$A = rb - sc, \; B = rc - sd, \; C = rd - sa, \; D = ra - sb. \tag{6}$$

Then

$$(A, B, C, D) \in S. \tag{7}$$

Proof:
$$\begin{aligned} A^3 + B^3 + C^3 + D^3 &= (r^3 - s^3)(a^3 + b^3 + c^3 + d^3) \\ &\quad - 3r^2s(b^2c + c^2d + d^2a + a^2b) \\ &\quad + 3rs^2(bc^2 + cd^2 + da^2 + ab^2) \\ &= (r^3 - s^3) \cdot 0 - 3r^2s \cdot s + 3rs^2 \cdot r = 0. \end{aligned}$$

Thus, $(A, B, C, D) \in S$ given by (5) and (6) may be constructed from the given quadruple $(a, b, c, d) \in S$, solving Part 2 of the problem.

Example: If $(a, b, c, d) = (3, 4, 5, -6)$, then $r = 274$, $s = 74$, $(A, B, C, D) = (726, 1814, -1866, 526)$.

Also solved by the proposer.

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