ON THE ENUMERATOR FOR SUMS OF THREE SQUARES

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1. INTRODUCTION

For each nonnegative integer n, $r_3(n)$ denotes the cardinal number of the set:

$$\{(x_1, x_2, x_3) \in \mathbb{Z}^3 | n = x_1^2 + x_2^2 + x_3^2\}.$$

We here propose to express r_3 in terms of simple divisor functions, defined as follows.

Definition: For each pair of positive integers i, n, with $i \leqslant 2$, $\delta_i(n)$ is defined by

$$\delta_i(n) = \sum_{\substack{d|n \\ d \equiv i \pmod 3}} (-1)^{(n/d)-1}.$$

Theorem 1: Let n denote an arbitrary positive integer.

(i) If $n = 3m^2$, for some positive integer m, then

$$r_3(n) = 2 + 6(-1)^n[\delta_2(n) - \delta_1(n)] + 12(-1)^n \sum_{i=1} (-1)^n [\delta_2(n - 3i^2) - \delta_1(n - 3i^2)].$$

(ii) If n is not of the form $3m^2$, then

$$r_3(n) = 6(-1)^n[\delta_2(n) - \delta_1(n)] + 12(-1)^n \sum_{i=1} (-1)^n[\delta_2(n - 3i^2) - \delta_1(n - 3i^2)].$$

In both statements (i) and (ii), summation for the sums indexed by i extends over all values of i for which the arguments of δ_1 and δ_2 are positive.

In §2, we prove this theorem. Our concluding remarks are concerned with comparison of the present representation of r_3 with the classical representation due to Dirichlet.

2. PROOF OF THEOREM 1

Our proof is predicated on the quintuple-product identity

$$\prod_1^\infty (1 - x^n)(1 - ax^n)(1 - a^{-1}x^{n-1})(1 - a^2x^{2n-1})(1 - a^{-2}x^{2n-1})$$
$$= \sum_{-\infty}^{\infty} x^{n(3n+1)/2}(a^{3n} - a^{-3n-1}), \tag{1}$$

which (as observed by Carlitz and Subbarao [1]) is derivable from the classical triple-product identity

$$\prod_1^\infty (1 - x^{2n})(1 + ax^{2n-1})(1 + a^{-1}x^{2n-1}) = \sum_{-\infty}^{\infty} x^{n^2}a^n. \tag{2}$$

Both identities are valid for each pair of complex numbers a, x such that $a \neq 0$ and $|x| < 1$. We shall also require the following classical identities associated with the names of Euler, Gauss, and Jacobi.

$$\prod_1^\infty (1 - x^{2n-1})(1 + x^n) = 1, \tag{3}$$

$$\prod_1^\infty (1 - x^{2n})(1 + x^{2n-1})^2 = \sum_{-\infty}^{\infty} x^{n^2}. \tag{4}$$

Identity (4) is an easy special case of (2) (simply set $a = 1$), but we list it separately to observe that the cube of its right side generates r_3.

In (1), let $a \to a^2$ and multiply the resulting identity by a to get:

$$\begin{aligned}(a - a^{-1}) &\prod_1^\infty (1 - x^n)(1 - a^2x^n)(1 - a^{-2}x^n)(1 - a^4x^{2n-1})(1 - a^{-4}x^{2n-1}) \\ &= a\sum_{-\infty}^{\infty} x^{n(3n+1)/2}a^{6n} - a^{-1}\sum_{-\infty}^{\infty} x^{n(3n+1)/2}a^{-6n} \\ &= a\prod_1^\infty (1 - x^{3n})(1 + a^6x^{3n-1})(1 + a^{-6}x^{3n-2}) \\ &\qquad - a^{-1}\prod_1^\infty (1 - x^{3n})(1 + a^{-6}x^{3n-1})(1 + a^6x^{3n-2}).\end{aligned} \tag{5}$$

Here we have used (2) to express the infinite series as infinite products. For the sake of brevity, put

$$F(a) = F(a, x) = \prod_1^\infty (1 - a^2x^n)(1 - a^{-2}x^n)(1 - a^4x^{2n-1})(1 - a^{-4}x^{2n-1}),$$

$$G(a) = G(a, x) = \prod_1^\infty (1 + a^6x^{3n-1})(1 + a^{-6}x^{3n-2}),$$

and

$$H(a) = G(a^{-1}).$$

Hence, (5) becomes

$$\prod_1^\infty (1 - x^n)(a - a^{-1})F(a) = \prod_1^\infty (1 - x^{3n})\{aG(a) - a^{-1}H(a)\}.$$

We now differentiate the foregoing identity with respect to a to get:

$$\begin{aligned}&\prod_1^\infty (1 - x^n)\{(1 + a^{-2})F(a) + (a - a^{-1})F'(a)\} \\ &= \prod_1^\infty (1 - x^{3n})\{G(a) + a^{-2}H(a) + aG'(a) - a^{-1}H'(a)\}.\end{aligned} \tag{6}$$

Sequentially, we use the technique of logarithmic differentiation to evaluate $G'(a)$ and $H'(a)$, substitute these evaluations into (6), let $a \to 1$ in the resulting identity, and finally cancel a factor of 2 to get:

$$\prod_1^\infty (1-x^n)^3(1-x^{2n-1})^2$$
$$= \prod_1^\infty (1-x^{3n})(1+x^{3n-1})(1+x^{3n-2})\left\{1+6\left(\sum_1^\infty \frac{x^{3n-1}}{1+x^{3n-1}} - \sum_1^\infty \frac{x^{3n-2}}{1+x^{3n-2}}\right)\right\}$$
$$= \prod_1^\infty (1-x^{3n})(1+x^{3n-1})(1+x^{3n-2})\left\{1+6\sum_1^\infty [\delta_2(n) - \delta_1(n)]x^n\right\}.$$

Now,

$$\prod_1^\infty \frac{(1-x^n)^3(1-x^{2n-1})^2}{(1-x^{3n})(1+x^{3n-1})(1+x^{3n-2})}$$
$$= \prod_1^\infty (1-x^n)^3(1-x^{2n-1})^3 \cdot \frac{(1+x^{3n})(1+x^{3n-1})(1+x^{3n-2})}{(1-x^{3n})(1+x^{3n-1})(1+x^{3n-2})}$$

[by Euler's identity (3)]

$$= \left\{\sum_0^\infty r_3(n)(-x)^n\right\} \cdot \prod_1^\infty \frac{1+x^{3n}}{1-x^{3n}}.$$

Hence,

$$\sum_0^\infty r_3(n)(-x)^n = \prod_1^\infty \frac{1-x^{3n}}{1+x^{3n}}\left\{1+6\sum_1^\infty [\delta_2(n) - \delta_1(n)]x^n\right\}$$
$$= \left\{1+2\sum_1^\infty (-x^3)^{n^2}\right\}\left\{1+6\sum_1^\infty [\delta_2(n) - \delta_1(n)]x^n\right\}.$$

Now, letting $x \to -x$, we have

$$\sum_0^\infty r_3(n)x^n = 1+2\sum_{m=1}^\infty x^{3m^2} + 6\sum_{n=1}^\infty (-1)^n[\delta_2(n) - \delta_1(n)]x^n$$
$$+ 12\sum_{n=1}^\infty (-1)^n x^n \sum_{i=1} (-1)^i[\delta_2(n-3i^2) - \delta_1(n-3i^2)].$$

[Here we adopt the convention that $\delta_i(k) = 0$ whenever $k < 0$, $i = 1, 2$.] Equating coefficients of like powers of x, we thus prove our theorem. [Note that $r_3(0) = 1$.]

CONCLUDING REMARKS

There is a somewhat complicated formula for $r_3(n)$ $[n \in \mathbb{Z}^+]$ due to Dirichlet. This is:

$$r_3(n) = \frac{16}{\pi} n^{1/2} \chi_2(n)K(-4n) \cdot \prod_{p^2|n}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{\tau-1}}+\frac{1}{p^\tau}\left(1-\left(\frac{p^{-2\tau}n}{p}\right)\frac{1}{p}\right)^{-1}\right),$$

where the definition of τ is $p^{2\tau}|n$, but $p^{2(\tau+1)} \nmid n$,

$$K(-4n) = \sum_{m=1}^\infty \left(\frac{-4n}{m}\right)\frac{1}{m},$$

Here, and above, $\left(\frac{-4n}{m}\right)$ is a Jacobi symbol. And

$$\chi_2(n) = \begin{cases} 0 & \text{if } 4^{-a}n \equiv 7 \pmod 8, \\ 2^{-a}, & \text{if } 4^{-a}n \equiv 3 \pmod 8, \\ 3.2^{-1-a}, & \text{if } 4^{-a}n \equiv 1, 2, 5, 6 \pmod 8, \end{cases}$$

and here the definition of a is $4^a | n$, but $4^{a+1} \nmid n$. This formula (among others) is given by Hua [2, pp. 215-216]. First of all, it is far from obvious that this expression for $r_3(n)$ is an integer, whereas our expressions of Theorem 1 are clearly integral. However, Dirichlet's formula permits an easy proof of the fact: $r_3(n) > 0$, if and only if, n is not of the form $4^a(8m + 7)$. At the moment, the author has not seen a way of deducing this fact from Theorem 1.

REFERENCES

1. L. Carlitz & M. V. Subbarao. "A Simple Proof of the Quintuple-Product Identity." *Proc. Amer. Math. Soc.* *32* (1972):42-44.
2. Loo-Keng Hua. *Introduction to Number Theory*. New York: Springer-Verlag, 1982.

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