AN ENTIRE FUNCTION THAT GIVES THE FIBONACCI NUMBERS AT THE INTEGERS

A. W. GOODMAN
University of South Florida, Tampa FL 33620
(Submitted August 1984)

1. INTRODUCTION

It is well known that $\Gamma(z)$ is an analytic function of z that gives $n!$ when $z = n + 1$. It is reasonable to look for a similar function for the Fibonacci numbers F_n. Several such functions are known (see Bunder [2] where further references are given), but the formula we will derive is more general than any of those obtained earlier.

To be specific, we are looking for an $F(z)$ with the following properties:

(a) $F(z)$ is an analytic function (perhaps entire),

(b) $F(z)$ is real valued for all real z,

(c) $F(n) = F_n$, the n^{th} Fibonacci number for all integers n,

(d) For z in the domain of analyticity we have

$$F(z + 2) = F(z + 1) + F(z). \tag{1}$$

It is clear that if $F(0) = F_0 = 0$ and $F(1) = F_1 = 1$, then equation (1) implies that $F(n) = F_n$ for every positive integer n. This follows immediately from the defining equations $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. In fact, this latter relation can be used to define the Fibonacci numbers for negative integers.

If $F(z)$ satisfies the functional equation (1), then so does each derivative $F^{(m)}(z)$, $m = 1, 2, \ldots$. This suggests that we try e^{Rz} as a solution, for some number R. When e^{Rz} is used in (1), we find that it is a solution if and only if e^R is a root of

$$x^2 = x + 1. \tag{2}$$

Using the standard notation for the roots of (2), we have

$$\alpha = \frac{1 + \sqrt{5}}{2}, \qquad \beta = \frac{1 - \sqrt{5}}{2}, \tag{3}$$

and hence

$$R = \ln \alpha \text{ or } R = \ln \beta = \ln|\beta| + (2q + 1)\pi i,\ q = 0, \pm 1, \pm 2, \ldots .$$

Using the linearity of (1) (see Spickerman [4]), it is clear that if p and q are integers, and C_1 and C_2 are arbitrary real numbers, then

$$f(z) = C_1 e^{z(\ln\alpha + 2p\pi i)} + C_2 e^{z(\ln|\beta| + (2q+1)\pi i)} \tag{4}$$

satisfies the functional equation (1). Now $f(z)$ is an entire function but it is not real valued for every real z. To remedy this defect, we consider

$$[f(z) + \overline{f(z)}]/2.$$

This function is not an analytic function, but if we replace z by x we obtain the real function

$$F(x) = C_1 e^{x \ln \alpha} \cos 2p\pi x + C_2 e^{x \ln|\beta|} \cos(2q + 1)\pi x. \tag{5}$$

If we now replace x by z in (5), we have a function that satisfies the conditions (a), (b), and (d). The initial conditions $F(0) = 0$ and $F(1) = 1$ force the selection $C_1 = 1/\sqrt{5}$ and $C_2 = -1/\sqrt{5}$. Then, finally, the function

$$F(z) = \frac{1}{\sqrt{5}}[e^{z \ln \alpha} \cos 2p\pi z - e^{z \ln|\beta|} \cos(2q + 1)\pi z] \tag{6}$$

has all of the properties (a), (b), (c), and (d) that we wish.

Equation (6) was given earlier by Spickerman [4] and is an entire function that gives the Fibonacci numbers for integral values of z.

2. THE MAIN THEOREM

Equation (6) gives a countable infinity of functions that satisfy the conditions (a), (b), (c), and (d), and we may ask if we now have all such functions. In fact, we shall soon see that (6) gives only a tiny portion of the functions that satisfy (a), (b), (c), and (d). We first observe that if α and β are the roots of (2) and m is an integer, then

$$G(z) = e^{z \ln \alpha} \sin 2m\pi z + e^{z \ln|\beta|} \sin(2m + 1)\pi z \tag{7}$$

satisfies the three conditions (a), (b), and (d). Further, $G(n) = 0$ for every integer n.

We now take linear combinations of the functions $F(z)$ and $G(z)$ defined by (6) and (7). To simplify the presentation, we impose a condition on the coefficients to ensure that we obtain entire functions.

Definition: We say that the real sequences $\{A_m\}$, $\{B_m\}$, $\{C_m\}$, and $\{D_m\}$ satisfy condition E if

$$\sum_{m=0}^{\infty} C_m = 1, \qquad \sum_{m=0}^{\infty} D_m = 1 \tag{8}$$

and

$$\sum_{m=0}^{\infty} A_m e^{mz}, \qquad \sum_{m=0}^{\infty} B_m e^{mz}, \qquad \sum_{m=0}^{\infty} C_m e^{mz}, \qquad \sum_{m=0}^{\infty} D_m e^{mz} \tag{9}$$

are all entire functions.

These are very weak restrictions. For example, (9) is trivially satisfied if all but a finite number of terms in each sequence are zero. The linearity of equation (1), and our earlier work, immediately give

Theorem 1: Let $\{A_m\}$, $\{B_m\}$, $\{C_m\}$, $\{D_m\}$ satisfy condition E, and let α and β be defined by (3). Then each one of the functions

$$F(z) = \frac{1}{\sqrt{5}} \sum_{m=0}^{\infty} C_m e^{z \ln \alpha} \cos 2m\pi z - \frac{1}{\sqrt{5}} \sum_{m=0}^{\infty} D_m e^{z \ln|\beta|} \cos(2m + 1)\pi z + \tag{10}$$

(continued)

$$+ \sum_{m=0}^{\infty} A_m e^{z \ln \alpha} \sin 2m\pi z + \sum_{m=0}^{\infty} B_m e^{z \ln|\beta|} \sin(2m + 1)\pi z$$

satisfies the conditions (a), (b), (c), and (d).

It is clear that (1) gives an uncountable infinity of suitable functions. We still have an uncountable infinity if we set all coefficients equal to zero except C_0, $C_1 = 1 - C_0$, D_0, and $D_1 = 1 - D_0$.

Do we have all such function? In other words, given a function with properties (a), (b), (c), and (d), is it one of the functions described in Theorem 1? This is an open problem.

The Fibonacci numbers satisfy many interesting relations, see, for example, Bachman [1, II:55-96], Vorob'ev [5], or Wall [6]. Many of these generalize, and we cite only a few here.

If $F(z)$ is any one of the uncountably many functions given in Theorem 1, then, for all z,

$$\sum_{k=0}^{N} F(z + k) = F(z + N + 2) - F(z + 1), \tag{11}$$

$$\sum_{k=1}^{N} F(z + 2k - 1) = F(z + 2N) - F(z), \tag{12}$$

and

$$\sum_{k=0}^{2N} (-1)^k F(z + k) = F(z + 2N - 1) - F(z - 2). \tag{13}$$

3. A GENERALIZATION

One natural generalization arises when we replace $F_{n+2} = F_{n+1} + F_n$ by

$$F_{n+2} = rF_{n+1} + sF_n$$

and impose the initial conditions $F_0 = a$ and $F_1 = b$. To extend the work of §1 and §2, we look for entire functions that are real on the real axis, give the generalized Fibonacci numbers at the positive integers, and satisfy the functional equation

$$F(z + 2) = rF(z + 1) + sF(z) \tag{14}$$

for all z. Here we restrict r, s, a, and b to be real. We preserve the basic notation of §2 and set

$$\alpha = \frac{r + \sqrt{r^2 + 4s}}{2}, \qquad \beta = \frac{r - \sqrt{r^2 + 4s}}{2}, \tag{15}$$

the two roots of

$$x^2 = rx + s. \tag{16}$$

[Compare this equation with equation (2).]

For simplicity, we assume that α and β are distinct real roots, and this implies that $r^2 + 4s > 0$. We also assume that $s \neq 0$ because, if $s = 0$, equation (14) reduces to $F(z + 1) = rF(z)$ for all z, and the generalized Fibonacci sequence is then a geometric sequence. If r and s are positive, then $\alpha > 0 > \beta$. We consider this case first.

Theorem 2: Suppose that $\alpha > 0 > \beta$, where α and β are given by (15), r and s are real numbers, and the sequences $\{A_m\}$, $\{B_m\}$, $\{C_m\}$, and $\{D_m\}$ satisfy condition E. Set

$$F(z) = \sum_{m=0}^{\infty} \frac{-a\beta + b}{\alpha - \beta} C_m e^{z \ln \alpha} \cos 2m\pi z + \sum_{m=0}^{\infty} \frac{a\alpha - b}{\alpha - \beta} D_m e^{z \ln|\beta|} \cos(2m+1)\pi z$$
$$+ \sum_{m=0}^{\infty} (A_m e^{z \ln \alpha} \sin 2m\pi z + B_m e^{z \ln|\beta|} \sin(2m+1)\pi z). \tag{17}$$

Then:

(a) $F(z)$ is an entire function;

(b) $F(z)$ is real on the real axis;

(c) $F(z)$ satisfies the functional equation (14);

(d) for all positive integers $F(n) = F_n$, the n^{th} generalized Fibonacci number defined by $F_0 = a$, $F_1 = b$, $F_{n+2} = rF_{n+1} + sF_n$, $n = 0, 1, 2, \ldots$.

We omit the proof because it follows the pattern set forth in §2. First, one shows that each individual term satisfies (14), and then one applies the linearity property. A simple computation shows that $F(0) = a$ and $F(1) = b$. Parker [3] obtained a simplified version of (17) in which only two of the coefficients are different from zero.

If $r > 0$ and $s < 0$, then $\alpha > \beta > 0$. In this case, we have

Theorem 3: Suppose that $\alpha > \beta > 0$ and the sequences $\{A_m\}$, $\{B_m\}$, $\{C_m\}$, and $\{D_m\}$ satisfy condition E. Set

$$F(z) = \sum_{m=0}^{\infty} \frac{-a\beta + b}{\alpha - \beta} C_m e^{z \ln \alpha} \cos 2m\pi z + \sum_{m=0}^{\infty} \frac{a\alpha - b}{\alpha - \beta} D_m e^{z \ln \beta} \cos 2m\pi z$$
$$+ \sum_{m=0}^{\infty} (A_m e^{z \ln \alpha} \sin 2m\pi z + B_m e^{z \ln \beta} \sin 2m\pi z). \tag{18}$$

Then $F(z)$ satisfies conditions (a), (b), (c), and (d) of Theorem 2.

The proof is similar to that of Theorem 2; thus, it is omitted here.

If $r < 0$ and $s < 0$, then $0 > \alpha > \beta$. In this case, we replace α and β by $|\alpha|$ and $|\beta|$, respectively, in (18). Further, $\cos 2m\pi z$ is replaced by $\cos(2m+1)\pi z$ and $\sin 2m\pi z$ is replaced by $\sin(2m+1)\pi z$. The details are left to the reader.

In each of the three cases, there is an uncountable infinity of functions, each satisfying the conditions (a), (b), (c), and (d).

4. CONCLUDING REMARKS

We return to the original Fibonacci sequence 0, 1, 1, 2, 3, 5, ... treated in §§1 and 2. If α and β are given by (3), then, as is well known,

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n). \tag{19}$$

This formula for F_n is called Binet's formula. If we replace the minus sign by a plus sign in (19), we obtain

$$L_n = \alpha^n + \beta^n. \tag{20}$$

These numbers L_n, $n = 0, 1, 2, \ldots$, are often called the Lucas numbers [5, 6]. Now $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for $n = 0, 1, 2, \ldots$. Consequently, Theorem 2 gives a set of uncountably many entire functions for the Lucas numbers. Indeed, set $a = 2$ and $b = 1$ in (17) to obtain

$$\frac{-a\beta + b}{\alpha - \beta} = 1 \quad \text{and} \quad \frac{a\alpha - b}{\alpha - \beta} = 1. \tag{21}$$

Then $F(n) = L_n$ for all n.

Finally, we note that Binet's formula can be extended to cover the generalized Fibonacci numbers treated in §3. Let r, s, a, b, α, and β be real numbers, where α and β are given by (15). If $F_0 = a$, $F_1 = b$, $F_{n+2} = rF_{n+1} + sF_n$, for $n = 0, 1, 2, \ldots$, then

$$F_n = \frac{-a\beta + b}{\alpha - \beta}\alpha^n + \frac{a\alpha - b}{\alpha - \beta}\beta^n, \text{ for } n = 0, 1, 2, \ldots . \tag{22}$$

Here, of course, we assume that $r^2 + 4s > 0$ so $\alpha \neq \beta$ and both α and β are real numbers. For brevity, we omit the discussion of the special cases (a) $\alpha = \beta$, (b) $\alpha = 0 > \beta$, and (c) $\alpha > \beta = 0$. In these last two cases, equation (16) gives $s = 0$. Hence, $F_{n+1} = rF_n$ and the sequence $\{F_n\}$ is a geometric sequence.

REFERENCES

1. Paul Bachman. *Niedere Zahlentheorie*. New York: Chelsea, 1968.
2. M. W. Bunder. "More Fibonacci Functions." *The Fibonacci Quarterly* **16** (1978):97-98.
3. Francis D. Parker. "A Fibonacci Function." *The Fibonacci Quarterly* **6** (1968):1-2.
4. W. R. Spickerman. "A Note on Fibonacci Functions." *The Fibonacci Quarterly* *8* (1970):397-401.
5. N. N. Vorob'ev. *Fibonacci Numbers*. New York: Blaisdell, 1961.
6. Charles R. Wall. *Selected Topics in Elementary Number Theory*. Columbia: University of South Carolina Press, 1971.

◆◇◆◇◆