DISCOVERING FIBONACCI IDENTITIES

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1. INTRODUCTION

One of the more appealing aspects of the Fibonacci sequence, and certainly the most appealing to the uninitiated, is the very large number of remarkable identities that can be found. Discussing identities with Vern Hoggatt several years ago, I pointed out that it was easy to discover new identities simply by varying the pattern of known identities and using inductive reasoning to guess new results. With characteristic enthusiasm, Vern immediately picked up on the idea and suggested that an appropriate paper be written. Shortly after returning home, I received a letter from Vern which began: "There are a surprising number of good ways of expanding the list of identities. Consider ... ." And the last sentence read: "At least some of this is sparkling new, and we are only using observation."

What follows is an account of some of the ideas we were sharing. They are not deep but, like Vern, I find them interesting. Of course, the ideas can be extended to more general recurrent sequences in obvious ways, but we restrict our attention here to the familiar Fibonacci and Lucas sequences defined by

$$F_i = \frac{\alpha^i - \beta^i}{\sqrt{5}} \quad \text{and} \quad L_i = \alpha^i + \beta^i, \tag{1}$$

where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, and i is an integer.

2. THE GENERAL IDEA

The identities

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1} \tag{2}$$

and

$$\sum_{i=1}^{n} L_i^2 = L_n L_{n+1} - 2 \tag{3}$$

are well known (see, for example, [4], p. 55]. Alternatively, for the Lucas sequence, one can easily obtain

$$\sum_{i=1}^{n} L^2 = \begin{cases} 5F_n F_{n+1} & n \text{ even}, \\ 5F_n F_{n+1} - 4 & n \text{ odd}. \end{cases} \tag{3'}$$

How might these be generalized? Well, sums of squares might be viewed as sums of terms of the *second degree* in Fibonacci and Lucas numbers. Thus, one might consider other such sums like, for example,

$$\sum_{i=1}^{n} F_i F_{i+1},\ \sum_{i=1}^{n} F_i F_{i+2},\ \ldots,\ \sum_{i=1}^{n} F_i F_{i+d}$$

and their Lucas counterparts or the mixed sums

$$\sum_{i=1}^{n} F_i L_{i+1},\ \sum_{i=1}^{n} F_i L_{i+2},\ \ldots,\ \sum_{i=1}^{n} F_i L_{i+d}.$$

One can now proceed formally, or with a little guessing, to obtain, for d any positive integer,

$$\sum_{i=1}^{n} F_i F_{i+d} = \begin{cases} F_n F_{n+d+1} & n \text{ even,} \\ F_n F_{n+d+1} - F_d & n \text{ odd,} \end{cases} \tag{4}$$

$$\sum_{i=1}^{n} L_i L_{i+d} = \begin{cases} 5F_n F_{n+d+1} & n \text{ even,} \\ 5F_n F_{n+d+1} - L_{d+3} & n \text{ odd,} \end{cases} \tag{5}$$

$$\sum_{i=1}^{n} F_i L_{i+d} = \begin{cases} F_n L_{n+d+1} & n \text{ even,} \\ F_n F_{n+d+1} - L_d & n \text{ odd,} \end{cases} \tag{6}$$

$$\sum_{i=1}^{n} L_i F_{i+d} = \begin{cases} F_n L_{n+d+1} & n \text{ even,} \\ F_n L_{n+d+1} - F_{d+3} & n \text{ odd,} \end{cases} \tag{7}$$

which, as one would expect, exhibit a pleasing symmetry.

The proofs are straightforward utilizing Binet's formulas (1) and the known identities (see [1] and [10])

$$F_{r+2s} - F_r = \begin{cases} F_s L_{r+s} & s \text{ even,} \\ L_s F_{r+s} & s \text{ odd,} \end{cases} \tag{8}$$

$$L_{r+2s} - L_r = \begin{cases} 5F_s F_{r+s} & s \text{ even,} \\ L_s L_{r+s} & s \text{ odd,} \end{cases} \tag{9}$$

$$F_{r+2s} + F_r = \begin{cases} L_s F_{r+s} & s \text{ even,} \\ F_s L_{r+s} & s \text{ odd,} \end{cases} \tag{10}$$

$$L_{r+2s} + L_r = \begin{cases} L_s L_{r+s} & s \text{ even} \\ 5F_s F_{r+s} & s \text{ odd.} \end{cases} \tag{11}$$

As an example of the proofs of (4)-(7), we prove (4). Since $\alpha\beta = -1$, $1 - \alpha^2 = -\alpha$, and $1 - \beta^2 = -\beta$, we have

$$\sum_{i=1}^{n} F_i F_{i+d} = \sum_{i=1}^{n} \left(\frac{\alpha^i - \beta^i}{\sqrt{5}}\right)\left(\frac{\alpha^{i+d} - \beta^{i+d}}{\sqrt{5}}\right)$$

(continued)

$$= \frac{\alpha^d(\alpha^2 - \alpha^{2n+2})}{5(1-\alpha^2)} + \frac{\beta^d(\beta^2 - \beta^{2n+2})}{5(1-\beta^2)} - \left(\frac{\alpha^d + \beta^d}{5}\right)\cdot \sum_{i=1}^{n} (-1)^i$$

$$= \frac{1}{5}\left\{\alpha^d(\alpha^{2n+1} - \alpha) + \beta^d(\beta^{2n+1} - \beta) - L_d \cdot \sum_{i=1}^{n}(-1)^i\right\}$$

$$= \frac{1}{5}\left\{\alpha^{2n+d+1} + \beta^{2n+d+1} - (\alpha^{d+1} + \beta^{d+1}) - L_d \cdot \sum_{i=1}^{n}(-1)^n\right\}.$$

Therefore, for n even,

$$\sum_{i=1}^{n} F_i F_{i+d} = \frac{1}{5}(L_{2n+d+1} - L_{d+1}) = F_n F_{n+d+1}$$

by (9). For n odd, we have

$$\sum_{i=1}^{n} F_i F_{i+d} = \frac{1}{5}(L_{2n+d+1} - L_{d+1} + L_d) = \frac{1}{5}(L_{2n+d+1} - L_{d-1})$$

$$= \frac{1}{5}(L_{2n+d+1} + L_{d+1} - 5F_d) = F_n F_{n+d+1} - F_d$$

by (11), since $L_{d+1} = L_d + L_{d-1}$ and $L_{d-1} + L_{d+1} = 5F_d$ for all d.
The other results are proved similarly.

3. THE IDENTITY $L_n^2 - 5F_n^2 = 4(-1)^n$

As a second example, we consider the identity

$$L_n^2 - 5F_n^2 = 4(-1)^n. \tag{12}$$

Again the terms on the left are of the second degree and we are led to consider expressions like

$$L_n^2 - 5F_{n+d}^2,\ L_nL_{n+d} - 5F_nF_{n+d},\ L_nF_{n+d} - F_nL_{n+d},\ L_nL_m - 5F_{n+d}F_{m+d},$$

and so on. As before, one can proceed either inductively or formally, or with a combination of both approaches, and it is a meta-theorem that we will not be disappointed. In fact, the following results can be exhibited. Let m, n, and d be integers. Then

$$L_nL_m - 5F_{n+d}F_{m+d} = \begin{cases} 5F_{-d}F_{m+n+d} + 2(-1)^n L_{m-n} & d \text{ even}, \\ L_{-d}L_{m+n+d} & d \text{ odd}, \end{cases} \tag{13}$$

$$L_nL_m - L_{n+d}L_{m+d} = \begin{cases} 5F_{-d}F_{m+n+d} & d \text{ even}, \\ L_{-d}L_{m+n+d} + 2(-1)^n L_{m-n} & d \text{ odd}, \end{cases} \tag{14}$$

$$F_nF_m - F_{n+d}F_{m+d} = \begin{cases} F_{-d}F_{m+n+d} & d \text{ even}, \\ \frac{1}{5}(L_{-d}L_{m+n+d} - 2(-1)^n L_{m-n}) & d \text{ odd}, \end{cases} \tag{15}$$

$$L_nF_m - L_{n+d}F_{m+d} = \begin{cases} F_{-d}L_{m+n+d} & d \text{ even}, \\ L_{-d}F_{m+n+d} - 2(-1)^n F_{m-n} & d \text{ odd}, \end{cases} \tag{16}$$

$$L_nF_m - L_{m+d}F_{n+d} = \begin{cases} F_{-d}L_{m+n+d} + 2(-1)^n F_{m-n} & d \text{ even}, \\ L_{-d}F_{m+n+d} & d \text{ odd}, \end{cases} \tag{17}$$

$$L_nL_{m+d} - 5F_nF_{m+d} = 2(-1)^n L_{m-n+d}, \tag{18}$$

$$L_nF_{m+d} - L_{m+d}F_n = 2(-1)^n F_{m-n+d}, \tag{19}$$

$$L_nL_m - 5F_{n-d}F_{m+d} = (-1)^n L_{-d}L_{m-n+d}, \tag{20}$$

$$L_nL_m - L_{n-d}L_{m+d} = 5(-1)^{n+1}F_{-d}F_{m-n+d}, \tag{21}$$

$$F_nF_m - F_{n-d}F_{m+d} = (-1)^{n+1}F_{-d}F_{m-n+d}, \tag{22}$$

$$L_nF_m - L_{n-d}F_{m+d} = (-1)^n F_{-d}L_{m-n+d}, \tag{23}$$

and

$$F_nL_m - L_{n-d}F_{m+d} = (-1)^{n+1}L_{-d}F_{m-n+d}. \tag{24}$$

Moreover, these identities, or the known identities,

$$F_{m+n+1} = F_mF_n + F_{m+1}F_{n+1}, \tag{25}$$

$$L_{m+n+1} = L_mF_n + L_{m+1}F_{n+1}, \tag{26}$$

and

$$5F_{m+n+1} = L_mL_n + L_{m+1}L_{n+1}, \tag{27}$$

suggest that we seek identities like (13)-(24) but with a plus sign on the left in place of the minus sign. Identities indeed exist and, somewhat surprisingly, are exactly the same as before but with the even and odd cases reversed. Thus, for example

$$F_nF_m + F_{n+d}F_{m+d} = \begin{cases} \frac{1}{5}(L_{-d}L_{n+m+d} - 2(-1)^n L_{m-n}) & d \text{ even}, \\ F_{-d}F_{n+m+d} & d \text{ odd}. \end{cases} \tag{28}$$

This should be compared with (15) above. Since this is the only change required, we refrain from listing the remaining counterparts to (13)-(24).

The proofs of (13)-(24) and their counterparts with the plus sign on the left-hand side all depend on Binet's formulas, identities (7)-(9) and equivalent identities obtained by replacing d by $-d$, and on the identities

$$F_{-n} = (-1)^{n-1}F_n \tag{29}$$

and

$$L_{-n} = (-1)^n L_n \tag{30}$$

for all n. As an example, we prove (14). We have

$$L_nL_m - L_{n+d}L_{m+d} = (\alpha^n + \beta^n)(\alpha^m + \beta^m) - (\alpha^{n+d} + \beta^{n+d})(\alpha^{m+d} + \beta^{m+d})$$

(continued)

$$= \alpha^{m+n} + \beta^{m+n} + (\alpha\beta)^n(\alpha^{m-n} + \beta^{m-n})$$

$$- \alpha^{m+n+2d} - \beta^{m+n+2d} - (\alpha\beta)^{n+d}(\alpha^{m-n} + \beta^{m-n})$$

$$= -(L_{m+n+2d} - L_{m+n}) + (-1)^n[1 - (-1)^d]L_{m-n}.$$

Thus, using (9), (29), and (30), we have, for d even,

$$L_nL_m - L_{n+d}L_{m+d} = -5F_dF_{m+n+d} = 5F_{-d}F_{m+n+d}$$

and, for d odd,

$$L_nL_m - L_{n+d}L_{m+d} = -L_dL_{m+n+d} + 2(-1)^nL_{m-n} = L_{-d}L_{m+n+d} + 2(-1)^nL_{m-n}$$

as claimed.

4. HIGHER-ORDER IDENTITIES

Casting about for other identities to treat in the same way,

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0 \tag{31}$$

and

$$F_{n+3}F_{n+4} - 2F_{n+2}F_{n+3} - 2F_{n+1}F_{n+2} + F_nF_{n+1} = 0 \tag{32}$$

were found in a paper by Hoggatt and Bicknell [6]. Note that (32) is already related to (31) in the manner of this paper, and one would expect such results as

$$L_{n+3}^2 - 2L_{n+2}^2 - 2L_{n+1}^2 + L_n^2 = 0, \tag{33}$$

$$F_{n+3}L_{n+3} - 2F_{n+2}L_{n+2} - 2F_{n+1}L_{n+1} + F_nL_n = 0, \tag{34}$$

$$F_{n+3}L_{m+3} - 2F_{n+2}L_{m+2} - 2F_{n+1}L_{m+1} + F_nL_m = 0, \tag{35}$$

and so on. Checking a bit further, I found that these and a good deal more are already known to hold. In [2], T. Brennan shows that

$$\sum_{r=0}^{n+1}(-1)^{r(r+1)/2}\begin{bmatrix} n+1 \\ r \end{bmatrix}x^{n+1-r} = 0 \tag{36}$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{F_nF_{n-1}\cdots F_{n-r+1}}{F_rF_{r-1}\cdots F_1}, \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = 1, \tag{37}$$

is the auxiliary equation for q_n, where q_n is the product of any n sequences satisfying the recurrence $\mu_{n+2} = \mu_{n+1} + \mu_n$.

For $n = 2$, (36) becomes

$$x^3 - 2x^2 - 2x + 1 = 0, \tag{38}$$

which implies the truth of (33)-(35) and all other such generalizations. For $n = 3$, (36) becomes

$$x^4 - 3x^3 - 6x^2 + 3x + 1 = 0, \tag{39}$$

which implies such identities as

$$F_{n+4}^3 - 3F_{n+3}^3 - 6F_{n+2}^3 + 3F_{n+1}^3 + F_n^3 = 0, \tag{40}$$

$$F_{n+4}F_{m+4}F_{p+4} - 3F_{n+3}F_{m+3}L_{p+3} - 6F_{n+2}F_{m+2}L_{p+2} + 3F_{n+1}F_{m+1}L_{p+1} + F_nF_mL_p = 0, \tag{41}$$

and so on. Those interested in these and similar matters should also see [3], [5], [7], [8], [9], and [11].

As a final example, we consider the well-known and elegant identity

$$F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3 \tag{42}$$

(see [12], p. 11). In the spirit of this paper, there are three immediate generalizations, and one has only to consider a few examples to guess the following:

$$L_{3n} = L_{n+1}F_{n+1}^2 + L_nF_n^2 - L_{n-1}F_{n-1}^2, \tag{43}$$

$$5F_{3n} = F_{n+1}L_{n+1}^2 + F_nL_n^2 - F_{n-1}L_{n-1}^2, \tag{44}$$

and

$$5L_{3n} = L_{n+1}^3 + L_n^3 - L_{n-1}^3. \tag{45}$$

For completeness, we prove each of (42)-(45). They are not difficult, but are a bit subtle, and it is easy to take a wrong turn. We make repeated use of (25), (26), and (27), above.

To prove (42), we use (25), and write

$$\begin{aligned}
F_{3n} &= F_{n-1+2n+1} \\
&= F_{n-1}F_{2n} + F_nF_{2n+1} \\
&= F_{n-1}F_{n-1+n+1} + F_nF_{n+n+1} \\
&= F_{n-1}(F_{n-1}F_n + F_nF_{n+1}) + F_n(F_n^2 + F_{n+1}^2) \\
&= F_{n-1}F_n(F_{n-1} + F_{n+1}) + F_n^3 + F_nF_{n+1}^2 \\
&= F_{n-1}(F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1}) + F_n^3 + F_nF_{n+1}^2 \\
&= F_{n-1}F_{n+1}^2 - F_{n-1}^3 + F_n^3 + F_nF_{n+1}^2 \\
&= F_{n+1}^2(F_{n-1} + F_n) + F_n^3 - F_{n-1}^3 \\
&= F_{n+1}^3 + F_n^3 - F_{n-1}^3
\end{aligned}$$

as claimed.

For (43), we use (26) and the formulas

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2 \quad \text{and} \quad F_{2n+1} = F_n^2 + F_{n+1}^2$$

from the proof of (42) to write

$$\begin{aligned}
L_{3n} &= L_{n-1+2n+1} \\
&= L_{n-1}F_{2n} + L_nF_{2n+1} \\
&= L_{n-1}(F_{n+1}^2 - F_{n-1}^2) + L_n(F_n^2 + F_{n+1}^2) \\
&= L_{n-1}F_{n+1}^2 - L_{n-1}F_{n-1}^2 + L_nF_n^2 + L_nF_{n+1}^2
\end{aligned}$$

(continued)

$$= F_{n+1}^2(L_{n-1} + L_n) + L_nF_n^2 - L_{n-1}F_{n-1}^2$$
$$= L_{n+1}F_{n+1}^2 + L_nF_n^2 - L_{n-1}F_{n-1}^2.$$

For (44), we use (27) and (26) to write

$$\begin{aligned}
5F_{3n} &= 5F_{n-1+2n+1}\\
&= L_{n-1}L_{2n} + L_nL_{2n+1}\\
&= L_{n-1}L_{n-1+n+1} + L_nL_{n+n+1}\\
&= L_{n-1}(L_{n-1}F_n + L_nF_{n+1}) + L_n(L_nF_n + L_{n+1}F_{n+1})\\
&= L_{n-1}^2F_n + L_{n-1}L_nF_{n+1} + L_n^2F_n + L_nL_{n+1}F_{n+1}\\
&= L_{n-1}^2(F_{n+1} - F_{n-1}) + L_{n-1}L_nF_{n+1} + L_n^2F_n + L_n(L_{n-1} + L_n)F_{n+1}\\
&= F_{n+1}(L_{n-1}^2 + 2L_{n-1}L_n + L_n^2) + L_n^2F_n - L_{n-1}^2F_{n-1}\\
&= F_{n+1}(L_{n-1} + L_n)^2 + L_n^2F_n - L_{n-1}^2F_{n-1}\\
&= F_{n+1}L_{n+1}^2 + F_nL_n^2 - F_{n-1}L_{n-1}^2.
\end{aligned}$$

Finally, to obtain (45), we use (26) and (27) to write

$$\begin{aligned}
5L_{3n} &= 5L_{n-1+2n+1}\\
&= 5(L_{n-1}F_{2n} + L_nF_{2n+1})\\
&= L_{n-1}\cdot 5F_{n-1+n+1} + L_n\cdot 5F_{n+n+1}\\
&= L_{n-1}(L_{n-1}L_n + L_nL_{n+1}) + L_n(L_n^2 + L_{n+1}^2)\\
&= L_{n-1}^2(L_{n+1} - L_{n-1}) + L_{n-1}L_nL_{n+1} + L_n^3 + L_nL_{n+1}^2\\
&= L_{n-1}^2L_{n+1} - L_{n-1}^3 + L_{n-1}L_nL_{n+1} + L_n^3 + (L_{n+1} - L_{n-1})L_{n+1}^2\\
&= L_{n+1}^3 + L_n^3 - L_{n-1}^3 + L_{n-1}L_{n+1}(L_{n-1} + L_n - L_{n+1})\\
&= L_{n+1}^3 + L_n^3 - L_{n-1}^3
\end{aligned}$$

as claimed.

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