SEQUENCES GENERATED BY MULTIPLE REFLECTIONS

IAN BRUCE
St. Peter's Collegiate School, Stonyfell, S. Australia
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1. We consider the situation of a light ray multiply reflected by a set of parallel glass plates in contact. The ray is assumed to be totally reflected or transmitted at any interface. A sequence is formed by considering the number of distinct ways a ray can be reflected n times before emerging. It is well known that this is the Fibonacci sequence if only two plates are present [1]. Several aspects of the general case for k plates have already been considered: Moser and Wyman [2] place a plane mirror behind the stack of plates, while Hoggatt and Junge [3] tackle the above situation. We will show how the enumerating matrices of [2] and [3] are related, and derive a procedure for evaluating the asymptotic form of the general sequence. In addition, some Fibonacci-like relations of the general sequence are shown.

2. We will restrict ourselves to the cases of two and three plates in this section, with generalizations being obvious to k plates. A scheme for counting the reflections of a given order is shown in Diagrams 1 and 2. A string of digits is used to enumerate the labelled interfaces at which reflections occur.

<u>2 plates</u>: (2, 3), (21, 31, 32), (212, 213, 312, 313, 323), ... (1)

<u>3 plates</u>: (2, 3, 4), (21, 31, 32, 41, 42, 43), (212, 213, 224, ...) (2)

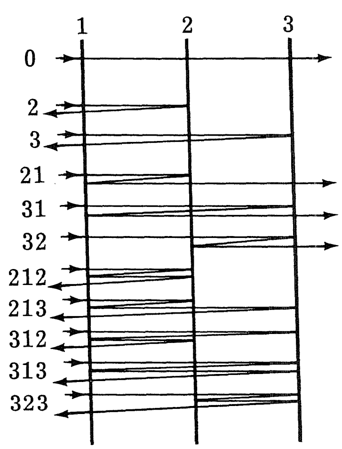

Diagram 1. Some of the labelled reflections from two sheets of glass.

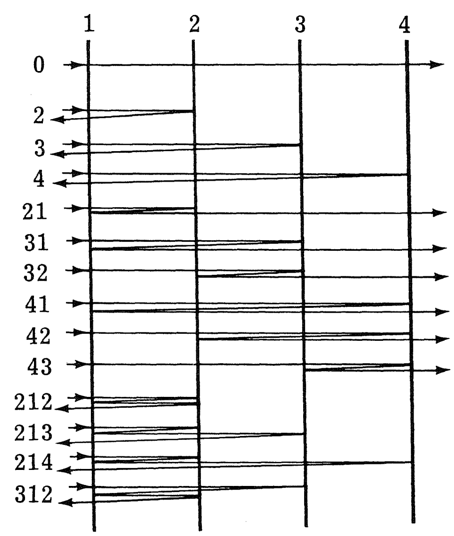

Diagram 2. Labelled reflections from three sheets of glass.

The reflections can now be shown without recourse to drawing them. All the reflections of a given order are placed in parentheses above. The number of reflections of a given order that end on the same final interface are now counted, and arranged in a sequence whose non-zero members are non-decreasing. The zeros arise, of course, because the ray must finally pass out through the first or last face.

The sequence that arises from (1) is:

$$0, 1, 1, 0, 1, 2, 0, 2, 3, 0, 3, 5, 0, 5, 8, 0, 8, 13, 0, 13, 21, 0, 21, 34, 0, \ldots, \tag{3}$$

which is seen to contain the Fibonacci sequence. The sequence that arises from (2) is:

$$0, 1, 1, 1, 0, 1, 2, 3, 0, 3, 5, 6, 0, 6, 11, 14, 0, 14, 25, 31, 0, \ldots . \tag{4}$$

Now, (3) is the sequence generated by the starting conditions:

$$r_0 = 0,\ r_1 = r_2 = 1, \tag{5}$$

together with the recurrence relations:

$$r_{3n} = 0,\ r_{3n+1} = r_{3n-1},\ r_{3n+2} = r_{3n-1} + r_{3n-2}, \quad \text{for } n \geqslant 1. \tag{6}$$

In the same way, (4) is produced by

$$r_0 = 0,\ r_1 = r_2 = r_3 = 1, \tag{7}$$

where

$$\begin{aligned} &r_{4n} = 0,\ r_{4n+1} = r_{4n-1},\ r_{4n+2} = r_{4n-1} + r_{4n-2}, \\ &r_{4n+3} = r_{4n-1} + r_{4n-2} + r_{4n-3}, \quad \text{for } n \geqslant 1. \end{aligned} \tag{8}$$

Some simple sequence properties are now listed for the sequence (2). These are all readily proven from the definition (8):

$$r_1 + r_5 + r_9 + \cdots + r_{4n+1} = r_{4n+2}; \tag{9}$$

$$r_3 + r_7 + r_{11} + \cdots + r_{4n+3} = r_{4n+6} - 2; \tag{10}$$

$$r_2 + r_6 + r_{10} + \cdots + r_{4n+2} = r_{4n+6} - r_{4n+2} - 1; \tag{11}$$

$$r_{4n}^2 + r_{4n+1}^2 + r_{4n+2}^2 + r_{4n+3}^2 = r_{2(4n+3)+1}. \tag{12}$$

In establishing (11), the following result is needed:

$$r_{4n+6} - r_{4n+2} = r_{4n+2} + r_{4n-3}. \tag{13}$$

We can use these partial sums to give the sum of all the reflections up to order n:

$$\sum_{i=1}^{4n} r_i = r_{4n-2} + 2 \cdot r_{4n+2} - r_{4n-6} - 2. \tag{14}$$

3. We consider the general case to obtain a procedure for evaluating terms like those on the right-hand side of (14). Note first that the non-zero terms

in the sequence can be generated in the following matrix notation:

$$\begin{bmatrix} r_{nk+1} \\ r_{nk+2} \\ \vdots \\ r_{(n+1)k-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix} \begin{bmatrix} r_{(n-1)k+1} \\ r_{(n-1)k+2} \\ \vdots \\ r_{nk-1} \end{bmatrix}, \tag{15}$$

or

$$\mathbf{r}_n = A\mathbf{r}_{n-1} = A^n\mathbf{r}_0, \tag{16}$$

easily by induction, where $\mathbf{r}_0$ is the starting conditions column vector. As is pointed out in [2], this approach can only be made viable by making use of the eigenvalues (λ) and their corresponding eigenvectors ($\mathbf{u}$) as follows:

Repeated application of A to the eigenvector $\mathbf{u}$ gives

$$A\mathbf{u} = \lambda\mathbf{u},\ A^2\mathbf{u} = \lambda^2\mathbf{u},\ \dots,\ A^n\mathbf{u} = \lambda_n\mathbf{u}. \tag{17}$$

The solution of (16) follows on expressing $\mathbf{r}_0$ as a linear combination of the eigenvectors of A. However, [2] considers the case with the mirror, which involves a different enumerating matrix. This means that all the reflections of odd order are unaffected by the mirror because they proceed to the left in any case, while a reflection of even order is added to the next odd order. The matrix that does this is A^2, where A is defined as in (15).

We now proceed to find the eigenvalues of A from the determinant of order k:

$$D_k(\lambda) = |A - \lambda I| = 0. \tag{18}$$

Now, [3] provides the useful recurrence relation:

$$D_k(\lambda) = (2\lambda^2 - 1)D_{k-2}(\lambda) - \lambda^4 D_{k-4}(\lambda). \tag{19}$$

If we assume a solution to (18) of the form $D_k(\lambda) = P^k$, where P is a polynomial in λ, independent of k, then we find that

$$D_k(\lambda) = c_1 a^k + c_2 b^k + c_3 a^k \cdot (-1)^k + c_4 b^k \cdot (-1)^k, \tag{20}$$

where

$$P = \pm(((2\lambda^2 - 1) \pm \Delta)/2)^{1/2} = \pm a,\ \pm b,$$

where a is the root with the positive discriminant and b that with the negative discriminant, while

$$\Delta = (1 - 4\lambda^2)^{1/2}. \tag{21}$$

The coefficients c ($i = 1, 2, 3, 4$), which are independent of k, can be found using the four characteristic equations of lowest order, i.e.,

$$D_0(\lambda) = 1,\ D_1(\lambda) = -\lambda + 1,\ D_2(\lambda) = \lambda^2 - \lambda - 1, \text{ and}$$
$$D_3(\lambda) = \lambda^3 + 2\lambda^2 + \lambda - 1, \tag{22}$$

as follows:

When k is even,

$$D_k(\lambda) = 0 = (c_1 + c_3)a^k + (c_2 + c_4)b^k, \tag{23}$$

leading to

$$(1 + 2\lambda - \Delta)/(1 + 2\lambda + \Delta) = ((2\lambda^2 - 1 - \Delta)/(2\lambda^2 - 1 + \Delta))^{k/2}, \tag{24}$$

on making use of $D_0(\lambda)$ and $D_2(\lambda)$.

We can readily solve (24) on making the substitutions

$$\lambda = \frac{1}{2} \sin \theta = t/(1 + t^2), \text{ where } t = \tan \theta/2, \tag{25}$$

giving:

$$t^{2k+1} = 1, \text{ with solutions } t = e^{\frac{\pm 2n\pi i}{2k+1}}, \quad n = 0, 1, \ldots, k. \tag{26}$$

Hence, the eigenvalues are given by

$$\lambda = \frac{1}{2} \sec(2n\pi/2k + 1), \; n = 1, 2, \ldots, k. \tag{27}$$

When k is odd, a similar argument leads to solving

$$t^{2k+1} = -1, \tag{28}$$

giving the eigenvalues:

$$\lambda = \frac{1}{2} \sec(2n + 1)\pi/2k + 1. \tag{29}$$

We are now in a position to evaluate (16), which we will briefly show for the case $k = 2$: From (27), the eigenvalues are

$$\lambda_1 = \frac{1}{2} \sec 2\pi/5 \quad \text{and} \quad \lambda_2 = \frac{1}{2} \sec 4\pi/5,$$

with the corresponding eigenvectors

$$\begin{pmatrix} 1 \\ \mathbf{t} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{t} \\ -1 \end{pmatrix}, \tag{30}$$

on writing $\mathbf{t} = \frac{1}{2} \sec 2\pi/5$.

On expressing $\mathbf{r}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in terms of the eigenvectors, and on using (16), we find:

$$\begin{pmatrix} r_{3n+1} \\ r_{3n+2} \end{pmatrix} = A^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{2\mathbf{t} + 1}{\mathbf{t} + 2} \cdot \mathbf{t}^{n-1} \cdot \begin{pmatrix} 1 \\ \mathbf{t} \end{pmatrix} + \frac{\mathbf{t} - 2}{\mathbf{t} + 2} \cdot \mathbf{t}^{-n+1} \cdot \begin{pmatrix} \mathbf{t} \\ -1 \end{pmatrix}; \tag{31}$$

$k \geqslant 2$ values are best tackled numerically, as the algebra becomes excessive.

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