ON A SECOND NEW GENERALIZATION OF THE FIBONACCI SEQUENCE

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A new perspective to the generalization of the Fibonacci sequence was introduced in [1]. Here, we take another step in the same direction. In [1] we studied the sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ defined by

$$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \beta_{n+1} + \beta_n, & (n \geqslant 0) \\ \beta_{n+2} = \alpha_{n+1} + \alpha_n, \end{cases} \tag{1}$$

where a, b, c, and d are fixed real numbers. We also utilized the generalization $\{F_i(a, b)\}_{i=0}^{\infty}$, where

$$\begin{cases} F_0(a, b) = a \\ F_1(a, b) = b & (n \geqslant 0) \\ F_{n+2}(a, b) = F_{n+1}(a, b) + F_n(a, b) \end{cases}$$

so that $F_n = F_n(0, 1)$, where $\{F_i\}_{i=0}^{\infty}$ is the Fibonacci sequence.

We shall study here the properties of the sequences for the scheme,

$$\begin{cases} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d, \\ \alpha_{n+2} = \alpha_{n+1} + \beta_n, & (n \geqslant 0) \\ \beta_{n+2} = \beta_{n+1} + \alpha_n, \end{cases} \tag{2}$$

where a, b, c, and d are fixed real numbers, and will conclude with a theorem, similar to [1]. Since the proofs of the results in this paper are similar to those in [1], we shall only list the results and eliminate the proofs.

Obviously when $a = b$ and $c = d$, the schemes from (2), as well as from (1), coincide with the Fibonacci sequence $\{F_i(a, b)\}_{i=0}^{\infty}$. The first few terms of the sequences from (2) are:

n	α_n	β_n
0	a	b
1	c	d
2	$b + c$	$a + d$
3	$b + c + d$	$a + c + d$
4	$a + b + c + 2d$	$a + b + 2c + d$
5	$2a + b + 2c + 3d$	$a + 2b + 3c + 2d$
6	$3a + 2b + 4c + 4d$	$2a + 3b + 4c + 4d$
7	$4a + 4b + 7c + 6d$	$4a + 4b + 6c + 7d$
8	$6a + 7b + 11c + 10d$	$7a + 6b + 10c + 11d$
9	$10a + 11b + 17c + 17d$	$11a + 10b + 17c + 17d$

Lemma 1: For every $k \geqslant 0$:

(a) $\alpha_{6k} + \beta_0 = \beta_{6k} + \alpha_0$;

(b) $\alpha_{6k+1} + \beta_1 = \beta_{6k+1} + \alpha_1$;

(c) $\alpha_{6k+2} + \alpha_0 + \beta_1 = \beta_{6k+2} + \beta_0 + \alpha_1$;

(d) $\alpha_{6k+3} + \alpha_0 = \beta_{6k+3} + \beta_0$;

(e) $\alpha_{6k+4} + \alpha_1 = \beta_{6k+4} + \beta_1$;

(f) $\alpha_{6k+5} + \beta_0 + \alpha_1 = \beta_{6k+5} + \alpha_0 + \beta_1$.

Lemma 2: For every $n \geqslant 0$:

(a) $\alpha_{n+2} = \sum_{i=0}^{n} \beta_i + \alpha_1$; (b) $\beta_{n+2} = \sum_{i=0}^{n} \alpha_i + \beta_1$.

Lemma 3: For every $n \geqslant 0$:

(a) $\sum_{i=0}^{6k} (\alpha_i - \beta_i) = \alpha_0 - \beta_0$;

(b) $\sum_{i=0}^{6k+1} (\alpha_i - \beta_i) = \alpha_0 - \beta_0 + \alpha_1 - \beta_1$;

(c) $\sum_{i=0}^{6k+2} (\alpha_i - \beta_i) = 2\alpha_1 - 2\beta_1$;

(d) $\sum_{i=0}^{6k+3} (\alpha_i - \beta_i) = -\alpha_0 + \beta_0 + 2\alpha_1 - 2\beta_1$;

(e) $\sum_{i=0}^{6k+4} (\alpha_i - \beta_i) = -\alpha_0 + \beta_0 + \alpha_1 - \beta_1$;

(f) $\sum_{i=0}^{6k+5} (\alpha_i - \beta_i) = 0$.

Lemma 4: For every $n \geqslant 0$:

$$\alpha_{n+2} + \beta_{n+2} = F_{n+1} \cdot (\alpha_0 + \beta_0) + F_{n+2} \cdot (\alpha_1 + \beta_1).$$

As in [1], we express the members of the sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ when $n \geqslant 0$, as follows:

$$\begin{cases} \alpha_n = \gamma_n^1 \cdot a + \gamma_n^2 \cdot b + \gamma_n^3 \cdot c + \gamma_n^4 \cdot d \\ \beta_n = \delta_n^1 \cdot a + \delta_n^2 \cdot b + \delta_n^3 \cdot c + \delta_n^4 \cdot d \end{cases}$$

It is interesting to note that Lemmas 5-7 have results identical to those found in [1] for the sequences $\{\gamma_n^1\}_{n=0}^{\infty}$, $\{\gamma_n^2\}_{n=0}^{\infty}$, etc., even though they are different sequences.

Lemma 5: For every $n \geqslant 0$:

(a) $\gamma_n^1 + \delta_n^1 = F_{n-1}$; (c) $\gamma_n^3 + \delta_n^3 = F_n$;

(b) $\gamma_n^2 + \delta_n^2 = F_{n-1}$; (d) $\gamma_n^4 + \delta_n^4 = F_n$.

Lemma 6: For every $n \geqslant 0$

(a) $\gamma_n^1 + \gamma_n^2 = \delta_n^1 + \delta_n^2$; (b) $\gamma_n^3 + \gamma_n^4 = \delta_n^3 + \delta_n^4$.

Lemma 7: For every $n \geqslant 0$:

(a) $\delta_n^1 = \gamma_n^2$; (e) $\gamma_n^3 = \gamma_{n+1}^2$;
(b) $\delta_n^2 = \gamma_n^1$; (f) $\gamma_n^4 = \gamma_{n+1}^1$;
(c) $\delta_n^3 = \gamma_n^4$; (g) $\delta_n^3 = \delta_{n+1}^2$;
(d) $\delta_n^4 = \gamma_n^3$; (h) $\delta_n^4 = \delta_{n+1}^1$.

Let ψ be the integer function defined for every $k \geqslant 0$ by:

τ	$\psi(6k + \tau)$
0	1
1	0
2	-1
3	-1
4	0
5	1

Obviously, for every $n \geqslant 0$,

$$\psi(n + 3) = -\psi(n). \tag{3}$$

Using the definition of the function ψ, the following are easily proved by induction.

Lemma 8: For every $n \geqslant 0$:

(a) $\gamma_n^1 = \delta_n^1 + \psi(n)$; (c) $\gamma_n^3 = \delta_n^3 + \psi(n + 4)$;
(b) $\gamma_n^2 = \delta_n^2 + \psi(n + 3)$; (d) $\gamma_n^4 = \delta_n^4 + \psi(n + 1)$

Lemma 9: For every $n \geqslant 0$:

(a) $\gamma_{n+2}^1 = \gamma_{n+1}^1 + \gamma_n^1 + \psi(n + 3)$; (d) $\gamma_{n+2}^3 = \gamma_{n+1}^3 + \gamma_n^3 + \psi(n + 1)$;
(b) $\gamma_{n+2}^2 = \gamma_{n+1}^2 + \gamma_n^2 + \psi(n)$; (e) $\gamma_{n+2}^4 = \gamma_{n+1}^4 + \gamma_n^4 + \psi(n + 4)$;
(c) $\gamma_n^1 = \gamma_n^2 + \psi(n)$; (f) $\gamma_n^3 = \gamma_n^4 + \psi(n + 4)$.

From Lemmas 5, 7, 8, and (3), we obtain the equations:

$$\gamma_n^1 = \delta_n^2 = \frac{1}{2}(F_{n-2} + \psi(n));$$

$$\gamma_n^2 = \delta_n^1 = \frac{1}{2}(F_{n-1} + \psi(n + 3));$$

$$\gamma_n^3 = \delta_n^4 = \frac{1}{2}(F_n + \psi(n + 4));$$

$$\gamma_n^4 = \delta_n^3 = \frac{1}{2}(F_n + \psi(n + 1)).$$

Theorem: For every $n \geqslant 0$:

$$\alpha_n = \frac{1}{2}\{(F_{n-1} + \psi(n))a + (F_{n-1} + \psi(n + 3))b + (F_n + \psi(n + 4))c + (F_n + \psi(n + 1))d\}$$

$$= \frac{1}{2}\{(a + b)F_{n-1} + (c + d)F_n + \psi(n)a + \psi(n + 3)b + \psi(n + 4)c + \psi(n + 1)d\}.$$

$$\beta = \frac{1}{2}\{(F_{n-1} + \psi(n + 3))a + (F_{n-1} + \psi(n))b + (F_n + \psi(n + 1))c + (F_n + \psi(n + 4))d\}$$

$$= \frac{1}{2}\{(a + b)F_{n-1} + (c + d)F_n + \psi(n + 3)a + \psi(n)b + \psi(n + 1)c + \psi(n + 4)d\}.$$

On the basis of what has been done in [1] and in this paper, one could be led to generalize and examine sequences of the following types

$$\begin{cases} \alpha_0 = a, \ \beta_0 = b, \ \alpha_1 = c, \ \beta_1 = d, \\ \alpha_{n+2} = p \cdot \beta_{n+1} + q \cdot \beta_n, & (n \geqslant 0) \\ \beta_{n+2} = t \cdot \alpha_{n+1} + s \cdot \alpha_n, \end{cases}$$

$$\begin{cases} \alpha_0 = a, \ \beta_0 = b, \ \alpha_1 = c, \ \beta_1 = d, \\ \alpha_{n+2} = p \cdot \alpha_{n+1} + q \cdot \beta_n, & (n \geqslant 0) \\ \beta_{n+2} = t \cdot \beta_{n+1} + s \cdot \alpha_n, \end{cases}$$

for the fixed real numbers p, q, t, and s.

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REFERENCE

1. K. Atanassov, L. Atanassova, & D. Sasselov. "A New Perspective to the Generalization of the Fibonacci Sequence." *The Fibonacci Quarterly* 23, no. 1 (1985):21-28.

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