ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications concerning ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$ 

PROBLEMS PROPOSED IN THIS ISSUE

B-580 Proposed by Valentina Bakinova, Rondout Valley, NY

What are the three smallest positive integers $d$ such that no Lucas number $L_n$ is an integral multiple of $d$?

B-581 Proposed by Antal Bege, University of Cluj, Romania

Prove that, for every positive integer $n$, there are at least $\lceil n/2 \rceil$ ordered 6-tuples $(a, b, c, x, y, z)$ such that

$$F_n = ax^2 + by^2 - cz^2$$

and each of $a, b, c, x, y, z$ is a Fibonacci number. Here $\lceil t \rceil$ is the greatest integer in $t$.

B-582 Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy

It is known that every positive integer $N$ can be represented uniquely as a sum of distinct nonconsecutive positive Fibonacci numbers. Let $f(N)$ be the number of Fibonacci addends in this representation, $a = (1 + \sqrt{5})/2$, and $\lfloor x \rfloor$ be the greatest integer in $x$. Prove that

$$f(\lfloor a^p \rfloor) = \lfloor (n + 1)/2 \rfloor \text{ for } n = 1, 2, \ldots .$$

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B-583 Proposed by Dorin Andrica, University of Cluj-Napoca, Romania

For positive integers \( n \) and \( s \), let

\[
S_{n,s} = \sum_{k=1}^{n} \binom{n}{k} k^s.
\]

Prove that \( S_{n,s+1} = n(S_{n,s} - S_{n-1,s}) \).

B-584 Proposed by Dorin Andrica, University of Cluj-Napoca, Romania

Using the notation of B-583, prove that

\[
S_{n+k,s} = \sum_{k=0}^{s} \binom{s}{k} S_{n,k} S_{n+s-k,k}.
\]

B-585 Proposed by Constantin Gonciulea & Nicolae Gonciulea, Triant College, Drobeta Turnu-Severin, Romania

For each subset \( A \) of \( \{1, 2, \ldots, n\} \), let \( r(A) \) be the number of \( j \) such that \( \{j, j+1\} \subseteq A \). Show that

\[
\sum_{A \subseteq X} 2^{r(A)} = F_{2n+1}.
\]

SOLUTIONS

Pattern for Squares

B-556 Proposed by Valentina Bakinova, Rondout Valley, NY

State and prove the general result illustrated by

\[
4^2 = 16, \quad 34^2 = 1156, \quad 334^2 = 111556, \quad 3334^2 = 11115556.
\]

Solution by Thomas M. Green, Contra Costa College, San Pablo, CA

Let \( D_n = 1 + 10 + 10^2 + \ldots + 10^{n-1} \). The general result

\[
(3D_n + 1)^2 = 10^n D_n + 5D_n + 1
\]

is proved by expanding the left member and observing that \( 9D_n = 10^n - 1 \).

Note: The quantity \( D_n \) has several other interesting properties:

(i) \( D_n = 111\ldots111 \) \((n \text{ ones})\)

(ii) \( D_n^3 = 123\ldots n \ldots 321 \) \((n = 1, \ldots, 9)\)

(iii) \( D_n/9 = 123456789 \)

(iv) \( (b-1)D_n = b^n - 1 \) \((b \text{ is your number base})\)

(v) The sequence \( D_1, D_1^2, D_2^2, \ldots, D_n^2 \), is Pascal's triangle (with suitable restrictions on carrying) and the sequences \( D_1^n, D_2^n, D_3^n, \ldots, D_n^n \), are Pascal-like triangles where each entry is the sum of the \( n \) entries above it.

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Imre Merényi, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Not True Any Year

B-557 Proposed by Imre Merényi, Cluj, Romania

Prove that there is no integer \( n \geq 2 \) such that
\[
F_{3n-6} F_{3n-3} F_{3n+3} F_{3n+6} - F_{n-2} F_{n-1} F_{n+1} F_{n+2} = 1985^2 + 1.
\]

Solution by J. Suck, Essen, Germany

Since \( F_{3k} \equiv 2F_k \pmod{3} \) [see, e.g., B-182, The Fibonacci Quarterly 8 (Dec. 1970), for the more general \( F_{pk} \equiv F_p F_k \pmod{p} \), \( p \) a prime], the left-hand side is congruent to \((2^n - 1)F_{n-2} F_{n-1} F_{n+1} F_{n+2}\), hence to 0. But the right-hand side is not whatever the year may be: if \( y \equiv 0, 1, 2 \), then \( y^2 + 1 \equiv 1, 2, 2 \), respectively, \( \mod{3} \).


Impossible Equation

B-558 Proposed by Imre Merényi, Cluj, Romania

Prove that there are no positive integers \( m \) and \( n \) such that
\[
F_{3m}^2 - F_{3n} - 4 = 0.
\]

Solution by L.A.G. Dresel, University of Reading, England

Since \( F_3 = 2 \) and \( F_6 = 8 \), we have \( F_{3n} \equiv 2 \pmod{4} \) when \( n \) is odd, and \( F_{3n} \equiv 0 \) when \( n \) is even. Now consider the equation \( F_{3m}^2 = F_{3n} + 4 \). Clearly \( n \) cannot be odd, since \( F^2 \equiv 2 \pmod{4} \) is not possible. However, if \( n \) is even, \( F_{3m} \equiv 4 \pmod{8} \) and this implies \( F_{3m} \equiv \pm 2 \pmod{8} \). Hence \( F_{3m} \) is even, so that \( m = 3k \), where \( k \) is an integer, and therefore \( F_{3m} = F_{12k} \), which is divisible by 8. This contradicts \( F_{3m} \equiv \pm 2 \pmod{8} \). Hence there are no integers \( m \) and \( n \) such that \( F_{3m}^2 - F_{3n} - 4 = 0 \).

We note that the above argument actually proves the slightly stronger result that there are no integers \( m \) and \( n \) such that \( F_{2m}^2 - F_{3n} - 4 = 0 \).

Also solved by Paul S. Bruckman, László Cseh, Piero Filipponi, L. Kuipers, Sahib Singh, Lawrence Somer, M. Wachtel, and the proposer.

Golden Mean Identity

B-559 Proposed by László Cseh, Cluj, Romania

Let \( a = (1 + \sqrt{5})/2 \). For positive integers \( n \), prove that
\[
[a + .5] + [a^2 + .5] + \cdots + [a^n + .5] = L_{n+2} - 2,
\]
where \([x]\) denotes the greatest integer in \( x \).

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Solution by J. Foster, Weber State College, Ogden, UT

Since \( L_k = \alpha^k + \beta^k \) and, for \( k \geq 2 \), \([.5 - \beta^k] = 0\),

\[
\sum_{k=1}^{n} [\alpha^k + .5] = \sum_{k=1}^{n} [L_k - \beta^k + .5] = \sum_{k=1}^{n} (L_k + [.5 - \beta^k]) = \sum_{k=0}^{n} L_k - L_0 + [.5 - \beta] = (L_{n+2} - 1) - 2 + 1 = L_{n+2} - 2.
\]


Another Greatest Integer Identity

B-560 Proposed by László Cseh, Cluj, Romania

Let \( a \) and \([x]\) be as in B-559. Prove that

\[
[aP_1 + .5] + [aP_2 + .5] + \ldots + [aP_n + .5]
\]

is always a Fibonacci number.

Solution by C. Georgiou, University of Patras, Greece

We have

\[
a^nP_n = \frac{\alpha^{2n} - (-1)^n}{\sqrt{5}} = P_{2n} + \frac{\beta^{2n} - (-1)^n}{\sqrt{5}}
\]

and since

\[
\left[ \frac{\beta^{2n} - (-1)^n}{\sqrt{5}} + .5 \right] = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}
\]

the given sum becomes

\[
P_2 + 1 + P_4 + P_6 + \ldots + P_{2n} = P_{2n+1}.
\]


Q-Matrix Identity

B-561 Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy

(i) Let \( Q \) be the matrix \( \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \). For all integers \( n \), show that

\[
Q^n + (-1)^nQ^{-n} = L_nI,
\]

where \( I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

(ii) Find a square root of \( Q \), i.e., a matrix \( A \) with \( A^2 = Q \).
Solution by Sahib Singh, Clarion University, Clarion, PA

(i) If $n = 0$, then $Q^0 + (Q^0)^{-1} = 2I = L_0I$.
For $n \geq 1$, it follows by mathematical induction that:

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\
F_n & F_{n-1} \end{bmatrix}$$

$$Q^{-n} = (-1)^n \begin{bmatrix} F_{n-1} & -F_n \\
-F_n & F_{n+1} \end{bmatrix}$$

Thus, $Q^n + (-1)^nQ^{-n} = \begin{bmatrix} L_n & 0 \\
0 & L_n \end{bmatrix} = L_nI$ for all $n \geq 1$.

Changing $n$ to $-n$, the above equation becomes:

$$Q^{-n} + (-1)^nQ^n = L_{-n}I \text{ or } (-1)^n[q^n + (-1)^nQ^{-n}] = (-1)^nL_nI,$$

so that

$$Q^n + (-1)^nQ^{-n} = L_nI \text{ for } n \leq -1.$$ 

Thus, the result holds for all integers.

(ii) Let a square root of $Q$ be denoted by $S$ where $S = \begin{bmatrix} a & b \\
c & d \end{bmatrix}$. Then $S^2 = Q$ yields

$$a^2 + bc = 1; (a + d)b = 1; (a + d)c = 1; bc + d^2 = 0.$$

Solving these equations, we conclude that

$$a = \frac{1 + b^2}{2b}; c = b; d = \frac{1 - b^2}{2b}, \text{ where } b \text{ satisfies}$$

$$5b^4 - 2b^2 + 1 = 0.$$ 

Thus, a square root of $Q$ is

$$\begin{bmatrix} \frac{1 + b^2}{2b} & b \\
b & \frac{1 - b^2}{2b} \end{bmatrix},$$

where $b$ is a complex number satisfying $5b^4 - 2b^2 + 1 = 0$, which can be solved by the quadratic formula using $x = b^2$.