EUCLIDEAN COORDINATES AS GENERALIZED FIBONACCI NUMBER PRODUCTS

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1. INTRODUCTION AND DEFINITIONS

In [2], it was shown how to obtain the coordinates of a point in (real) three-dimensional Euclidean space as triple products of Fibonacci numbers.

This was achieved as a development of two-dimensional ideas involving complex numbers, though the three-dimensional extension was devoid of any dependence on complex numbers.

Here, we wish to enlarge these notions to more general recurrence-generated number sequences and then to generalize our result to n-dimensional Euclidean space. To accomplish this objective, we will need to introduce a symbol $G(\ell, m, n)$, originally defined in [2] in relation to Fibonacci numbers only. This symbol represents a number with three components which may be regarded as the coordinates of a point with respect to three rectangular Cartesian axes, x, y, and z, i.e., as Cartesian or "Euclidean" coordinates.

First, we define the recurrence sequence $\{U_n\}$ by

$$U_{n+2} = pU_{n+1} - qU_n,\ U_0 = 0,\ U_1 = 1 \qquad (n \geqslant 0), \tag{1.1}$$

where p and q are generally integers.

Next, for positive integers ℓ, m, n, let

$$\begin{cases} G(\ell + 2, m, n) = pG(\ell + 1, m, n) - qG(\ell, m, n) \\ G(\ell, m + 2, n) = pG(\ell, m + 1, n) - qG(\ell, m, n) \\ G(\ell, m, n + 2) = pG(\ell, m, n + 1) - qG(\ell, m, n) \end{cases} \tag{1.2}$$

with

$$\begin{cases} G(0, 0, 0) = (a, a, a),\ G(1, 0, 0) = (b, 0, 0),\ G(0, 1, 0) \\ \qquad = (0, b, 0), \\ G(0, 0, 1) = (0, 0, b),\ G(1, 1, 0) = p(b, b, 0),\ G(1, 0, 1) \\ \qquad = p(b, 0, b), \\ G(0, 1, 1) = p(0, b, b),\ G(1, 1, 1) = p^2(b, b, b) \end{cases} \tag{1.3}$$

a and b being integers.

2. PROPERTIES OF $G(\ell, m, n)$

Inductive proofs, with appeal to (1.1)-(1.3), readily establish the following (cf. [2]):

$$G(\ell, 0, 0) = U_\ell G(1, 0, 0) - qU_{\ell-1}G(0, 0, 0) \tag{2.1}$$

$$G(\ell, 1, 0) = U_\ell G(1, 1, 0) - qU_{\ell-1}G(0, 1, 0) \tag{2.2}$$

$$G(\ell, m, 0) = U_m G(\ell, 1, 0) - qU_{m-1}G(\ell, 0, 0) \tag{2.3}$$

$$G(\ell, 0, 1) = U_\ell G(1, 0, 1) - qU_{\ell-1}G(0, 0, 1) \tag{2.4}$$
$$G(\ell, 1, 1) = U_\ell G(1, 1, 1) - qU_{\ell-1}G(0, 1, 1) \tag{2.5}$$
$$G(\ell, m, 1) = U_m G(\ell, 1, 1) - qU_{m-1}G(\ell, 0, 1) \tag{2.6}$$
$$G(\ell, m, n) = U_n G(\ell, m, 1) = qU_{n-1}G(\ell, m, 0) \tag{2.7}$$

Then,

$$\begin{aligned} G(\ell, m, n) &= U_n\{U_m G(\ell, 1, 1) - qU_{m-1}G(\ell, 0, 1)\} - qU_{n-1}\{U_m G(\ell, 1, 0) \\ &\qquad - qU_{m-1}G(\ell, 0, 0)\} \text{ by (2.3), (2.6), and (2.7)} \\ &= U_m U_n G(\ell, 1, 1) - qU_{m-1}U_n G(\ell, 0, 1) - qU_m U_{n-1}G(\ell, 1, 0) \\ &\qquad + q^2 U_{m-1}U_{n-1}G(\ell, 0, 0) \\ &= U_m U_n\{p^2 U_\ell(b, b, b) - pqU_{\ell-1}(0, b, b)\} \qquad (2.8) \\ &\qquad - qU_{m-1}U_n\{pU_\ell(b, 0, b) - qU_{\ell-1}(0, 0, b)\} \\ &\qquad - qU_m U_{n-1}\{pU_\ell(b, b, 0) - qU_{\ell-1}(0, b, 0)\} \\ &\qquad + q^2 U_{m-1}U_{n-1}\{U_\ell(b, 0, 0) \\ &\qquad - qU_{\ell-1}(a, a, a)\} \text{ by (2.1), (2.2), (2.4),} \\ &\qquad \text{and (2.5).} \end{aligned}$$

Further,

$$\begin{aligned} U_\ell U_{m+1}U_{n+1} &= U_\ell(pU_m - qU_{m-1})(pU_n - qU_{n-1}) \quad \text{by (1.1)} \\ &= p^2 U_\ell U_m U_n - pqU_\ell U_m U_{n-1} - pqU_\ell U_{m-1}U_n + q^2 U_\ell U_{m-1}U_{n-1} \end{aligned} \tag{2.9}$$

with similar expressions for $U_{\ell+1}U_m U_{n+1}$ and $U_{\ell+1}U_{m+1}U_n$.

Comparing (2.8) and (2.9), we see that the right-hand side of (2.9) contains precisely those coefficients in (2.8) of coordinate sets with b in the first position, i.e., in the x-direction. Missing is the term in $U_{\ell-1}U_{m-1}U_{n-1}$.

Similar remarks apply to $U_{\ell+1}U_m U_{n+1}$ for b in the second position, and to $U_{\ell+1}U_{m+1}U_n$ for b in the third position, of a coordinate set.

Accordingly, we have established that

$$\begin{aligned} G(\ell, m, n) = (&p^2 bU_\ell U_{m+1}U_{n+1} - q^3 aU_{\ell-1}U_{m-1}U_{n-1}, \\ &p^2 bU_{\ell+1}U_m U_{n+1} - q^3 aU_{\ell-1}U_{m-1}U_{n-1}, \\ &p^2 bU_{\ell+1}U_{m+1}U_n - q^3 aU_{\ell-1}U_{m-1}U_{n-1}). \end{aligned} \tag{2.10}$$

Equation (2.10) gives the cooedinates of a point in three-dimensional Euclidean space in terms of numbers of the sequence $\{U_n\}$.

When $p = 1$, $q = -1$, $b = 1$, $a = 0$ in (1.1), we obtain the result for Fibonacci numbers F_n given in [2], namely,

$$G(\ell, m, n) = (F_\ell F_{m+1}F_{n+1}, F_{\ell+1}F_m F_{n+1}, F_{\ell+1}F_{m+1}F_n). \tag{2.11}$$

Setting $p = 2$, $q = -1$, $b = 1$, $a = 0$ in (1.1), we have the *Pell numbers* P_n for which (2.10) becomes

$$G(\ell, m, n) = (4P_\ell P_{m+1}P_{n+1}, 4P_{\ell+1}P_m P_{n+1}, 4P_{\ell+1}P_{m+1}P_n). \tag{2.12}$$

Before concluding this section we observe that, say, (2.6) may be expressed in an alternative form as

$$G(\ell, m, 1) = U_\ell G(1, m, 1) - qU_{\ell-1}(0, m, 1). \qquad (2.6)'$$

3. HIGHER-DIMENSIONAL SPACE

Suppose we now extend the definitions in (1.1)-(1.3) to n dimensions in a natural way as follows. (The use of n here is not to be confused with its use in a different context in the symbol G in the previous section.)

For the n variables ℓ_i ($i = 1, 2, \ldots, n$), we define

$$\begin{cases} G(\ell_1 + 2, \ell_2, \ell_3, \ldots, \ell_n) = pG(\ell_1 + 1, \ell_2, \ell_3, \ldots, \ell_n) - qG(\ell_1, \ell_2, \ell_3, \ldots, \ell_n) \\ G(\ell_1, \ell_2 + 2, \ell_3, \ldots, \ell_n) = pG(\ell_1, \ell_2 + 1, \ell_3, \ldots, \ell_n) - qG(\ell_1, \ell_2, \ell_3, \ldots, \ell_n) \\ \text{---} \\ G(\ell_1, \ell_2, \ell_3, \ldots, \ell_n + 2) = pG(\ell_1, \ell_2, \ell_3, \ldots, \ell_n + 1) - qG(\ell_1, \ell_2, \ell_3, \ldots, \ell_n) \end{cases} \tag{3.1}$$

with

$$\begin{cases} G(0, 0, 0, \ldots, 0) = (a, a, a, \ldots, a) \\ G(1, 1, 1, \ldots, 1) = (b, b, b, \ldots, b) \\ G(\text{---------------}) = p^k(\text{-------------}) \end{cases} \tag{3.2}$$

in which G(-----) contains $k + 1$ 1's and $n - (k + 1)$ 0's, and (-----) contains $k + 1$ b's and $n - (k + 1)$ 0's, in corresponding positions.

Mutatis mutandis, similar but more complicated results to those obtained in the previous section now apply to (3.1) and (3.2).

In particular, the result corresponding to (2.10) is

$$\begin{aligned} G(\ell_1, \ell_2, \ell_3, \ldots, \ell_n) = (&p^{n-1}bU_{\ell_1}U_{\ell_2+1}U_{\ell_3+1} \cdots U_{\ell_n+1} + U, \\ &p^{n-1}bU_{\ell_1+1}U_{\ell_2}U_{\ell_3+1} \cdots U_{\ell_n+1} + U, \\ &\text{--------------------------------} \\ &p^{n-1}bU_{\ell_1+1}U_{\ell_2+1}U_{\ell_3+1} \cdots U_{\ell_n} + U) \end{aligned} \tag{3.3}$$

where, for visual and notational ease, we have written

$$U = (-q)^n aU_{\ell_1-1}U_{\ell_2-1}U_{\ell_3-1} \cdots U_{\ell_n-1}. \tag{3.4}$$

Clearly, (3.3) may represent the coordinates of a point in n-dimensional Euclidean space in terms of the numbers of the sequence $\{U_n\}$.

For Fibonacci numbers, $U = 0$, and (3.3) reduces to

$$\begin{aligned} G(\ell_1, \ell_2, \ell_3, \ldots, \ell_n) = (&F_{\ell_1}F_{\ell_2+1}F_{\ell_3+1} \cdots F_{\ell_n+1}, \ldots, \\ &F_{\ell_1+1}F_{\ell_2+1}F_{\ell_3+1} \cdots F_{\ell_n}). \end{aligned} \tag{3.5}$$

Likewise, for Pell numbers, $U = 0$ also, and (3.3) becomes

$$\begin{aligned} G(\ell_1, \ell_2, \ell_3, \ldots, \ell_n) = (&2^{n-1}P_{\ell_1}P_{\ell_2+1}P_{\ell_3+1} \cdots P_{\ell_n+1}, \ldots, \\ &2^{n-1}P_{\ell_1+1}P_{\ell_2+1}P_{\ell_3+1} \cdots P_{\ell_n}). \end{aligned} \tag{3.6}$$

It does not appear that any useful geometrical applications of an elementary nature can be deduced from the above results.

Harman [2] noted that if, in his case for Fibonacci numbers, the three expressions in (1.2) are combined, then the value of $G(\ell + 2, m + 2, n + 2)$ is given by the sum of the values of the symbol G at the eight vertices of the cube diagonally below that point. Similar comments apply to our more general expressions (1.2) with corresponding observations for the extension to n dimensions entailed in (3.1) in connection with the 2^n vertices of a "hypercube."

By this statement, we mean that when, say, $n = 3$, (1.2) gives

$$\begin{aligned} G(\ell + 2, m + 2, n + 2) = {} & p^3G(\ell + 1, m + 1, n + 1) - p^2q\{G(\ell + 1, m + 1, n) \\ & + G(\ell + 1, m, n + 1) + G(\ell, m + 1, n + 1)\} \\ & + pq^2\{G(\ell + 1, m, n) + G(\ell, m + 1, n) \\ & + G(\ell, m, n + 1)\} - q^3G(\ell, m, n). \end{aligned} \tag{3.7}$$

In the case of Fibonacci numbers, $p^3 = -p^2q = pq^2 = -q^3 = 1$. For Pell numbers, $p^3 = 8$, $-p^2q = 4$, $pq^2 = 2$, $-q^3 = 1$.

4. CONCLUDING REMARKS

Consider briefly now the two-dimensional aspect of the results in the preceding section, i.e., the case $n = 2$. (Evidently, when $n = 1$, we merely get the numbers U_n strung out on the number axis.)

Writing $\ell_1 = \ell$, $\ell_2 = m$, we find that the truncated forms corresponding to (3.1)-(3.7) are, respectively,

$$\begin{cases} G(\ell + 2, m) = pG(\ell + 1, m) - qG(\ell, m), \\ G(\ell, m + 2) = pG(\ell, m + 1) - qG(\ell, m), \end{cases} \tag{4.1}$$

with

$$G(0, 0) = (a, a),\ G(1, 0) = (b, 0),\quad G(0, 1) = (0, b),\ G(1, 1) = p(b, b), \tag{4.2}$$

whence:

$$G(\ell, m) = (pbU_\ell U_{m+1} + aU_{\ell-1}U_{m-1},\ pbU_{\ell+1}U_m + aU_{\ell-1}U_{m-1}), \tag{4.3}$$

$$G(\ell, m) = (F_\ell F_{m+1},\ F_{\ell+1}F_m) \quad \text{for } \{F_n\}, \tag{4.5}$$

$$G(\ell, m) = (2P_\ell P_{m+1},\ 2P_{\ell+1}P_m) \quad \text{for } \{P_n\}, \tag{4.6}$$

$$\begin{aligned} G(\ell + 2, m + 2) = {} & p^2G(\ell + 1, m + 1) - pqG(\ell + 1, m) \\ & - pqG(\ell, m + 1) + q^2G(\ell, m). \end{aligned} \tag{4.7}$$

Obvious simplifications of (4.7) apply for Fibonacci and Pell numbers.

Some of the above results, for Fibonacci numbers in the real Euclidean plane, should be compared with the corresponding results in the complex (Gaussian) plane obtained in [2]. The present authors [5] have studied the consequences in the complex plane of a natural generalization of the material in [2]. Harman [2], when advancing the innovatory features of his approach, acknowledges the earlier work of [1] and [3], and relates his work to theirs. It might be noted in passing that the introductory comments on quaternions in [3] have been investigated by other authors, e.g., [4]. One wonders whether an application of quaternions to extend the above theory on complex numbers might be at all fruitful.

From the structure provided by the complex Fibonacci numbers, some interesting classical identities involving products are derivable ([2] and [5]). Hopefully, these might give a guide to identities involving triple products of Fibonacci numbers, as conjectured in [2], and products in more general recurrence-generated number systems, as herein envisaged.

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