A NOTE ON MOESSNER'S PROCESS

CALVIN T. LONG
Washington State University, Pullman, WA 99163
(Submitted February 1985)

1. INTRODUCTION

According to Moessner's Theorem [3], [6], the k^{th} powers of the positive integers can be generated in the following interesting way. Delete every k^{th} integer from the sequence of positive integers, form a new sequence by taking partial sums of the original altered sequence, delete every $(k - 1)^{\text{st}}$ entry from the sequence of partial sums, and so on. After $k - 1$ steps, this process terminates with the deletion of the sequence of k^{th} powers. For example, for $k = 3$, we have

$$\begin{array}{cccccccccccccc}
1 & 2 & \cancel{3} & 4 & 5 & \cancel{6} & 7 & 8 & \cancel{9} & 10 & 11 & \cancel{12} & \ldots \\
1 & \cancel{3} & & 7 & \cancel{12} & & 19 & \cancel{27} & & 37 & \cancel{48} & & \ldots \\
\cancel{1} & & & \cancel{8} & & & \cancel{27} & & & \cancel{64} & \ldots & &
\end{array}$$

Note that we can think of the process terminating when we delete the single element at the bottom vertex of each small triangular array. A more general result due to I. Paasche [4] is that, if $\{k_i\}$ is a sequence of nonnegative integers, if the sequence

$$k_1,\ 2k_1 + k_2,\ 3k_1 + 2k_2 + k_3,\ \ldots \tag{1}$$

is deleted from the sequence of positive integers, if the sequence of partial sums is formed, and so on, the process terminates with the sequence

$$1^{k_1},\ 2^{k_1}1^{k_2},\ 3^{k_1}2^{k_2}1^{k_3},\ \ldots .$$

For example, if $k_i = 1$ for all i, the numbers deleted are the triangular numbers, and we obtain

$$\begin{array}{cccccccccccc}
\cancel{1} & \cancel{2} & \cancel{3} & 4 & 5 & \cancel{6} & 7 & 8 & 9 & \cancel{10} & 11 & \ldots \\
 & \cancel{2} & & 6 & \cancel{11} & & 18 & 26 & \cancel{35} & & 46 & \ldots \\
 & & & \cancel{6} & & & 24 & \cancel{50} & & & 96 & \ldots \\
 & & & & & & \cancel{24} & & & & 120 & \ldots \\
 & & & & & & & & & & \cancel{120} & \ldots
\end{array}$$

and

$$1 = 1^1,\ 2 = 2^1 \cdot 1^1,\ 6 = 3^1 \cdot 2^1 \cdot 1^1,\ 24 = 4^1 \cdot 3^1 \cdot 2^1 \cdot 1^1$$

and

$$120 = 5^1 \cdot 4^1 \cdot 3^1 \cdot 2^1 \cdot 1^1.$$

Of course, this is more neatly written as

$$1 = 1!,\ 2 = 2!,\ 6 = 3!,\ 24 = 4!,\ 120 = 5!$$

If we start the Moessner process by deleting the triangular numbers

$$\frac{n(n+1)}{2} = \binom{n+1}{2}$$

we generate the factorials—a truly remarkable result!

It is natural to ask what happens if we commence the process by deleting the terms of other well-known sequences—say the Fibonacci or Lucas numbers, the square numbers, the binomial coefficients

$$\binom{n+k-1}{k}$$

for fixed k, the terms of a geometric progression $\{ar^{n-1}\}$ for positive integers a and $r > 1$, and other sequences the reader might think of. We might also ask what happens if the k's in (1) above are in some well-known sequence. Both of these questions are addressed in what follows. The interested reader will also want to consult [1], [2], [5], and [7].

2. AN INVERSION THEOREM

Let $f(n)$ be any increasing positive integer valued function whose successive values, $f(1)$, $f(2)$, $f(3)$, ..., we want to delete from the sequence of positive integers to initiate Moessner's process. To determine the products generated, it is necessary to determine the nonnegative integers k_i, $i \geq 1$, such that

$$f(1) = k_1, \quad f(2) = 2k_1 + k_2, \quad f(3) = 3k_1 + 2k_2 + k_3, \ \ldots,$$

i.e., such that

$$f(n) = \sum_{i=1}^{n} (n + 1 - i)k_i, \ n \geq 1. \tag{2}$$

Of course, the condition that the k's be nonnegative has implications for the growth rate of $f(n)$. Thus,

$$k_1 \leq k_1 + k_2 \leq k_1 + k_2 + k_3 \leq \cdots, \tag{3}$$

and so

$$f(1) \leq f(2) - f(1) \leq f(3) - f(2) \leq \cdots . \tag{4}$$

This will force some adjustments later on, but does not affect the following inversion theorem.

Theorem 1: Formulas (2) hold with

$$k_1 = f(1), \ k_2 = f(2) - 2f(1)$$

and

$$k_i = f(i) - 2f(i-1) + f(i-2), \text{ for } i \geq 3.$$

That is to say,

$$f(n) = nf(1) + (n-1)[f(2) - 2f(1)] + \sum_{i=3}^{n} (n+1-i)[f(i) - 2f(i-1) + f(i-2)]. \tag{5}$$

Proof: Clearly, $k_1 = f(1)$ and $k_2 = f(2) - 2k_1 = f(2) - 2f(1)$. Moreover, for $j \geqslant 2$,

$$f(j) - f(j-1) = \sum_{i=1}^{j} (j + 1 - i) - \sum_{i=1}^{j-1} (j - i)k_i = \sum_{i=1}^{j} k_i,$$

and hence, for $j \geqslant 3$,

$$\begin{aligned} k_j = \sum_{i=1}^{j} k_i - \sum_{i=1}^{j-1} k_i &= f(j) - f(j-1) - [f(j-1) - f(j-2)] \\ &= f(j) - 2f(j-1) + f(j-2) \end{aligned}$$

as claimed.

We now apply Theorem 1 to some interesting sequences, making sure at the same time that (2) and (4) are satisfied.

3. THE FIBONACCI SEQUENCE

If F_n denotes the n^{th} Fibonacci number, then

$$F_{i+1} - F_i = F_{i-1},$$

so the sequence of differences is nondecreasing for $i \geqslant 1$. Since $F_3 \geqslant 2F_2$, we may set $f(n) = F_{n+1}$ and the Moessner process will apply. Also, from Theorem 1, we have

$$k_1 = F_2 = 1,\ k_2 = F_3 - 2F_2 = 0,$$

and

$$k_i = F_{i+1} - 2F_i + F_{i-1} = F_{i-3}$$

for $i \geqslant 3$. Thus, from (5), we have

$$F_{n+1} = n + \sum_{i=3}^{n} (n + 1 - i)F_{i-3} \tag{6}$$

and if we delete the numbers 1, 2, 3, 5, 8, 13, ... from the sequence of positive integers, the Moessner process generates products with the exponents

$$1,\ 0,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots\ .$$

That is, the products generated are

$$1^1,\ 2^1 1^0,\ 3^1 2^0 1^0,\ 4^1 3^0 2^0 1^1,\ 5^1 4^0 3^0 2^1 1^1,\ 6^1 5^0 4^0 3^1 2^1 1^2,\ \ldots\ .$$

4. THE LUCAS SEQUENCE

There is a little difficulty with the Lucas sequence $\{L_n\}$ because of (4). Thus,

$$L_{i+1} - L_i = L_{i-1}$$

and the sequence of differences only increases for $i \geqslant 2$. Also, if we attempt to set $f(i) = L_{i+1}$ as for the Fibonacci sequence, then

$$f(2) = L_3 = 4 \ngeqslant 6 = 2L_2 = 2f(1).$$

This difficulty, however, can be overcome by a slight artifice. Consider the function $f(n) = n$ for $1 \leqslant n \leqslant 2$, and $f(n) = L_{n-1}$ for $n \geqslant 3$. Here the differences are nondecreasing for $i \geqslant 1$ and $f(2) \geqslant 2f(1)$. For this sequence, we have

$$k_1 = f(1) = 1,\ k_2 = f(2) - 2f(1) = 2 - 2 \cdot 1 = 0,$$
$$k_3 = f(3) - 2f(2) + f(1) = 3 - 2 \cdot 2 + 1 = 0,$$
$$k_4 = f(4) - 2f(3) + f(2) = 4 - 2 \cdot 3 + 2 = 0,$$

and for $i \geqslant 5$,

$$k_i = f(i) - 2f(i-1) + f(i-2) = L_{i-1} - 2L_{i-2} + L_{i-3} = L_{i-5}.$$

Thus, from (5), for $n \geqslant 4$, we have

$$\begin{aligned} L_{n-1} &= nk_1 + (n-1)k_2 + (n-2)k_3 + (n-3)k_4 + \sum_{i=5}^{n}(n+1-i)k_i \\ &= n + \sum_{i=5}^{n}(n+1-i)L_{i-5}, \end{aligned} \tag{7}$$

and, if we begin the Moessner process by deleting 1, 2, 3, 4, 7, 11, 18, ..., the exponents in the generated products are 1, 0, 0, 0, 2, 1, 3, 4, 7, ...

5. THE GENERAL SECOND-ORDER RECURRENCE

Consider the general second-order recurrence defined by $g_1 = c$, $g_2 = d$, and $g_{n+2} = ag_{n+1} + bg_n$ for $n \geqslant 1$, where a, b, c, and d are positive integers with $d \geqslant 2c$. The first few terms of $\{g_n\}$ are

$$g_1 = c,\ g_2 = d,\ g_3 = ad + bc,\ g_4 = a^2d + abc + bd,\ \ldots .$$

Now define the sequence $\{k_i\}$ by

$$\begin{aligned} k_1 &= g_1 = c, \\ k_2 &= g_2 - 2g_1 = d - 2c, \\ k_i &= g_i - 2g_{i-1} + g_{i-2}, \text{ for } i \geqslant 3, \end{aligned} \tag{8}$$

so that the k_i satisfy Theorem 1 for all $i \geqslant 1$. Then, deleting the sequence $\{g_i\}$ from the sequence of positive integers in the Moessner process generates products whose exponents are successive terms of the sequence $\{k_i\}$. In addition to the above, note that

$$k_3 = ad + bc - 2d + c,\quad k_4 = a^2d + abc + bd - 2ad - 2bc + d,$$

and that

$$k_n = ak_{n-1} + bk_{n-2}, \text{ for } n \geqslant 5.$$

We may also ask what sequence $\{f(i)\}$ should be deleted from the sequence of integers to start a Moessner process that generates products where the expontnes are the sequence $\{g_i\}$. We must determine $f(n)$ such that

$$f(n) = \sum_{i=1}^{n}(n+1-i)g_i. \tag{9}$$

It turns out that the desired function $f(n)$ may be defined by the following second-order, nonlinear recurrence.

$$f(1) = c,\ f(2) = 2c + d,$$
$$f(n + 1) = af(n) + bf(n - 1) - nac + (n + 1)c + nd,\ n \geqslant 2.$$

To see this, we note that

$$f(1) = c = g_1 \quad \text{and} \quad f(2) = 2c + d = 2g_1 + g_2.$$

Now assume that (9) holds for $n = k - 1$ and $n = k$ for some fixed $k \geqslant 2$. Then,

$$\begin{aligned}
f(k + 1) &= af(k) + bf(k - 1) - kac + (k + 1)c + kd \\
&= a\sum_{i=1}^{k}(k + 1 - i)g_i + b\sum_{i=1}^{k-1}(k - i)g_i - kac + (k + 1)c + kd \\
&= akg_1 + a\sum_{i=2}^{k}(k + 1 - i)g_i + b\sum_{i=1}^{k-1}(k - i)g_i - kac + (k + 1)c + kd \\
&= akc + a\sum_{j=2}^{k}(k + 1 - j)g_j + b\sum_{j=2}^{k}(k + 1 - j)g_{j-1} \\
&\qquad - kac + (k + 1)c + kd \\
&= \sum_{j=2}^{k}(k + 1 - j)(ag_j + bg_{j-1}) + (k + 1)c + kd \\
&= \sum_{i=1}^{k+1}(k + 2 - i)g_i.
\end{aligned}$$

This completes the induction.

Incidentally, it now follows from Theorem 1 that

$$g_i = f(i) - 2f(i - 1) + f(i - 2),\ i \geqslant 3. \tag{10}$$

6. SOME OTHER INTERESTING SEQUENCES

If we start the Moessner process by deleting terms in the arithmetic progression $\{a + (n - 1)d\}$, where $d \geqslant a$ in view of (4), it follows from Theorem 1 that the exponents in the generated products are

$$k_1 = a,$$
$$k_2 = (a + d) - 2a = d - a,$$

and $$k_i = [a + (i - 1)d] - 2[a + (i - 2)d] + [a + (i - 3)d] - 0,$$

for $i \geqslant 3$. Thus, the generated products are simply

$$1^a,\ 2^a1^{d-a},\ 3^a2^{d-a},\ 4^a3^{d-a},\ \ldots .$$

If, instead of starting Moessner's process by deleting the terms of an arithemtic progression, we desire that the k's (i.e., the exponents in the resulting products) be in arithmetic progression, we must delete the successive terms of the sequence $\{f(n)\}$, where

$$f(n) = \sum_{i=1}^{n} (n + 1 - i)[a + (i - 1)d] = \binom{n + 1}{2}a + \binom{n + 1}{3}d. \tag{11}$$

In Section 1 we saw that interesting results were obtained if we began the Moessner process by deleting the triangular numbers

$$\frac{n(n + 1)}{2} = \binom{n + 1}{2}, \quad n \geqslant 1.$$

This naturally raises the question of deleting the binomial coefficients

$$\binom{n + k + 1}{k} \text{ for any fixed integer } k \geqslant 2.$$

Of course, the result follows from Theorem 1 with

$$f(n) = \binom{n + k - 1}{k}.$$

We have

$$k_1 = f(1) = \binom{1 + k - 1}{k} = \binom{k}{k} = \binom{k - 2}{k - 2},$$

$$k_2 = f(2) - 2f(1) = \binom{2 + k - 1}{k} - 2\binom{1 + k - 1}{k} = k - 1 = \binom{k - 1}{k - 2},$$

and, for $i \geqslant 3$,

$$k_i = \binom{i + k - 1}{k} - 2\binom{i - 1 + k - 1}{k} + \binom{i - 2 + k - 1}{k} = \binom{i + k - 3}{k - 2}.$$

That is,

$$\binom{n + k - 1}{k} = \sum_{i=1}^{n} (n + 2 - i)\binom{i + k - 3}{k - 2}. \tag{12}$$

Thus, if we delete the sequence $\left\{\binom{n + 2}{3}\right\}_{n \geqslant 1}$, we generate the products

$$1^{\binom{1}{1}},\ 2^{\binom{1}{1}}1^{\binom{2}{1}},\ 3^{\binom{1}{1}}2^{\binom{2}{1}}1^{\binom{3}{1}},\ \ldots\ .$$

If we delete the sequence $\left\{\binom{n + 3}{4}\right\}_{n \geqslant 1}$, we generate the products

$$1^{\binom{2}{2}},\ 2^{\binom{2}{2}}1^{\binom{3}{2}},\ 3^{\binom{2}{2}}2^{\binom{3}{2}}1^{\binom{4}{2}},\ \ldots,$$

and so on.

Finally, we consider the case when $\{f(n)\}$, the sequence of deleted numbers, is the geometric progression $\{ar^{n-1}\}$ with a and r positive integers and $r \geqslant 2$; and also the case where $k_i = ar^{i-1}$.

If $f(n) = ar^{n-1}$, we are starting Moessner's process deleting the terms of a geometric progression and we have from Theorem 1 that the exponents in the generated products are

$$k_1 = f(1) - a,$$

$$k_2 = f(2) - 2f(1) = ar - 2a = a(r - 2),$$

and, for $i \geqslant 3$,

$$\begin{aligned} k_i &= f(i) - 2f(i-1) + f(i-2) \\ &= ar^{i-1} - 2ar^{i-2} + ar^{i-3} \\ &= ar^{i-3}(r^2 - 2r + 1) \\ &= ar^{i-3}(r-1)^2, \end{aligned}$$

again a geometric progression with common ratio r after the first two terms. In any case, we also have that

$$ar^{n-1} = na + (n-1)a(r-2) + a(r-1)^2 \sum_{i=3}^{n} r^{i-3}. \tag{13}$$

If, on the other hand, $k_i = ar^{i-1}$ for $i \geqslant 1$, we must begin Moessner's process by deleting the successive terms of the sequence $\{f(n)\}$, where

$$\begin{aligned} f(n) &= a\sum_{i=1}^{n}(n+i-1)r^{i-1} = a\sum_{i=0}^{n-1}(n-i)r^i \\ &= a\sum_{i=0}^{n-1}\frac{(n-i)r^i r^{n-i-1}}{r^{n-i-1}} = ar^{n-1}\sum_{i=0}^{n-1}(n-i)x^{n-i-1}, \end{aligned}$$

where $x = r^{-1}$. Thus,

$$\begin{aligned} f(n) &= r^{n-1} \cdot \frac{d}{dx}\sum_{i=0}^{n-1}x^{n-i}\bigg|_{x=r^{-1}} = ar^{n-1} \cdot \frac{d}{dx}\frac{x - x^{n+1}}{1-x}\bigg|_{x=r^{-1}} \\ &= \frac{a(r^{n+1} - r + n - rn)}{(r-1)^2}, \end{aligned} \tag{14}$$

and the Moessner process yields the products

$$1^a,\ 2^a \cdot 1^{ar},\ 3^a \cdot 2^{ar} \cdot 1^{ar^2},\ \ldots .$$

ACKNOWLEDGMENT

The author wishes to thank the referee for several suggestions that helped to improve the presentation of this paper.

REFERENCES

1. C. Long. "On the Moessner Theorem on Integral Powers." *Amer. Math. Monthly* *73* (1966):846-851.
2. C. Long. "Strike It Out—Add It Up." *Math. Gazette* *66* (1982):173-177.
3. A. Moessner. "Eine bemerkung uber die potenzen der naturlichen zahlen." *Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B.* (1951):29.
4. I. Paasche. "Eine verallgemeinerung des Moessnerschen satzes." *Compositio Math.* *12* (1955):263-270.
5. I. Paasche. "Ein zahlenteoretisch-logarithmischer rechenstab." *Math. u. Naturwiss. Unterr.* *6* (1953/54):26-28.
6. O. Person. "Beweis des Moessnerschen satzes." *Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B.* (1951):31-34.
7. J. Slater. "Strike It Out—Some Exercises." *Math. Gazette* *67* (1983):288-290.

◆◇◆◇◆