ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-406 *Proposed by R. A. Melter, Long Island University, Southampton, NY and I. Tomescu, University of Bucharest, Romania*

Let A_n denote the set of points on the real line with coordinates 1, 2, ..., n. If $F(n)$ denotes the number of pairwise noncongruent subsets of A_n, then prove

$$F(n) = \begin{cases} 2^{n-2} + 2^{n/2} - 1 & \text{for } n \text{ even,} \\ 2^{n-2} + 3 \cdot 2^{(n-3)/2} - 1 & \text{for } n \text{ odd.} \end{cases}$$

H-407 *Proposed by Paul S. Bruckman, Fair Oaks, CA*

Find a closed form for the infinite product

$$\prod_{n=0}^{\infty} \frac{(5n+2)(5n+3)}{(5n+1)(5n+4)}. \tag{1}$$

H-408 *Proposed by Robert Shafer, Berkeley, CA*

a) Define $u_0 = 3$, $u_1 = 0$, $u_2 = 2$, and $u_{n+1} = u_{n-1} + u_{n-2}$ for all integers n.

b) In addition, let $w_0 = 3$, $w_1 = 0$, $w_2 = -2$, and $w_{n+1} = -w_{n-1} + w_{n-2}$ for all integers n.

Prove: $u_p \equiv w_p \equiv 0 \pmod{p}$ and $u_{-p} \equiv -w_{-p} \equiv -1 \pmod{p}$, where p is a prime number.

SOLUTIONS

Here's the Limit!

H-383 *Proposed by Clark Kimberling, Evansville, IN (Vol. 23, no. 1, February 1985)*

For any $x > 0$, let

$$c_1 = 1, \quad c_2 = x, \quad \text{and} \quad c_n = \frac{1}{n}\sum_{i=1}^{n} c_i c_{n-i} \quad \text{for } n = 3, 4, \ldots .$$

Prove or disprove that there exists $y > 0$ such that $\lim_{n \to \infty} y^n c_n = 1$.

Solution by Paul S. Bruckman, Fair Oaks, CA

We form the generating function

$$u = f(z, x) = \sum_{n=1}^{\infty} c_n z^n, \tag{1}$$

assumed valid for some disk of convergence $C: |z| < r$ (z complex). Under this assumption,

$$\lim_{n \to \infty} c_{n+1}/c_n = 1/r. \tag{2}$$

Note that from the defining recurrence and the condition $x > 0$ it follows that all c_n's are positive. Within C, the series defining f represents an analytic function of z, hence may be differentiated term by term. Thus,

$$u' = \sum_{n=1}^{\infty} nc_n z^{n-1} = 1 + 2xz + \sum_{n=3}^{\infty} nc_n z^{n-1} = 1 + 2xz + \sum_{n=3}^{\infty} z^{n-1} \sum_{k=1}^{n-1} c_k c_{n-k}$$

$$= 1 + 2xz - z + \sum_{n=2}^{\infty} z^{n-1} \sum_{k=1}^{n-1} c_k c_{n-k}$$

$$= 1 - (1 - 2x)z + \sum_{k=1}^{\infty} c_k z^{k-1} \sum_{n=1}^{\infty} c_n z^n,$$

or

$$u' = u^2/z + 1 - az, \tag{3}$$

where

$$a = 1 - 2x. \tag{4}$$

Note also the conditions

$$f(0, x) = 0, \quad f'(0, x) = 1. \tag{5}$$

For reasons which will become clear subsequently, we make the initial restriction $x \neq 1/2$, i.e., $a \neq 0$. We make the fortuitous substitutions:

$$u = -zv'/v, \text{ where } v = g(z, x), \tag{6}$$

$$v = w/\sqrt{\theta}, \tag{7}$$

$$w = h(\theta, x), \tag{8}$$

$$\theta = 2bz, \tag{9}$$

$$b = \sqrt{a}. \tag{10}$$

Upon transformation, we obtain the following differential equation:

$$w'' + (-1/4 + 1/2b\theta + 1/4\theta^2)w = 0, \tag{11}$$

where differentiation in (11) is with respect to θ.

Equation (11) is a special case of Whittaker's Equation, given in 13.1.31 of [1] in the following (paraphrased) modified form:

$$w'' + \left(-\frac{1}{4} + \frac{k}{\theta} + \frac{1/4 - \mu^2}{\theta^2}\right)w = 0. \tag{12}$$

Equation (12) possesses multiple-valued solutions, but we are not concerned with these; there exists a single-valued solution of (12) which meets all the necessary criteria. This is given by Whittaker's Function (13.1.32 [1]):

$$M_{k,\mu}(\theta) = e^{-1/2\theta}\,\theta^{1/2+\mu}M(1/2 + \mu - k,\ 1 + 2\mu,\ \theta), \tag{13}$$

where $M(A, B, Z)$ is the Kummer (or Confluent Hypergeometric) function defined by

$$M(A, B, Z) = \sum_{n=0}^{\infty}\frac{(A)_n}{(B)_n}\cdot\frac{z^n}{n!}, \tag{14}$$

using Pochhammer's notation: $(s)_n = s(s + 1)(s + 2)\ldots(s + n - 1)$.

Note (11) is obtained from (12) by setting $k = 1/2b$, $\mu = 0$; the restriction $b \neq 0$ now becomes evident. We therefore obtain the solution of (11):

$$g(\theta) = e^{-1/2\theta}\,\theta^{1/2}\,M(c,\ 1,\ \theta), \tag{15}$$

where

$$c = \frac{b - 1}{2b}. \tag{16}$$

Thus, using (6) and (7),

$$v = g(z, x) = e^{-bz}M(c,\ 1,\ 2bz). \tag{17}$$

To check that the boundary conditions in (5) are satisfied, we note that

$$\frac{d}{dZ}M(A, B, Z) = \frac{A}{B}M(A + 1,\ B + 1,\ Z);$$

hence,

$$v' = e^{-bz}\{2bcM(C + 1,\ 2,\ 2bz) - bM(c,\ 1,\ 2bz)\},$$

$$v'' = e^{-bz}\{2b^2c(c + 1)M(c + 2,\ 3,\ 2bz) - 4b^2cM(c + 1,\ 2,\ 2bz) + b^2M(c,\ 1,\ 2bz)\}.$$

Since $M(A, B, 0) = 1$, thus,

$$g(0, x) = 1,\ g'(0, x) = 2bc - b = -1),$$

$$g''(0, x) = 2b^2c(c + 1) - 4b^2c + b^2 = b^2(2c \quad - 2c + 1)$$
$$= 1/2(b - 1)^2 - b(b - 1) + b^2 = 1/2(b^2 + 1) = 1 - x.$$

Using (6), $f(0, x) = \dfrac{0(1)}{1} = 0$. Also,

$$u' = \frac{-v(zv'' + v') + z(v')^2}{v^2} = \frac{-zv''}{v} - \frac{v'}{v} + z(v'/v)^2;$$

hence,

$$f'(0, x) = 0(1 - x)/1 + 1/1 + 0(-1/1)^2 = 1.$$

Thus, the boundary conditions are satisfied.

Using the divverential expression for v' obtained above and simplifying, we obtain as the (single-valued) solution of (3):

$$u = f(z, x) = bz - (b - 1)z \cdot \frac{M(c + 1, 2, 2bz)}{M(c, 1, 2bz)}, \tag{18}$$

provided $x \neq 0$.

If $0 < x < 1/2$, then $0 < a < 1$, $0 < b < 1$, $c < 0$. In this situation, it is known that $M(c, 1, 2bz)$ has a zero z_0 of minimum modulus $|z_0| > 0$. Since e^{-bz} cannot vanish for any values of z, thus z_0 is also a zero of g. Hence, from (6), z_0 is a simple pole of f; moreover, there are no other singularities of f with smaller modulus than z_0. It follows from (2) that

$$\lim_{n \to \infty} c_{n+1}/c_n = |z_0|^{-1}. \tag{19}$$

From a known result in analysis (Ex. 68.1 [2]):

$$\lim_{n \to \infty} c_n^{1/n} = |z_0|^{-1}. \tag{20}$$

Now (20) implies

$$\lim_{n \to \infty} |z_0|^n c_n = 1. \tag{21}$$

Hence, if $0 < x < 1/2$, the original conjecture is true, with $y = |z_0|$. If $x > 1/2$, then $a < 0$, $b = ik = i\sqrt{-a}$, say, and $c = 1/2 + i/2k$. Less seems to be known about the zeros of $M(A, B, Z)$ when A, B, and Z are complex, in particular of the function

$$M\left(\frac{1}{2} + \frac{i}{2k}, 1, 2ikz\right);$$

it seems likely, however, that, in this case as well, there exists a nonzero zero of this function, which leads to (21), as before. Certainly, the numerical evidence suggests this conclusion; namely, that the conjecture is true for $x > 1/2$.

Only the case $x = 1/2$ remains to be investigated. In this case, $a = b = 0$, but as $x \to 1/2^-$, $c \to -\infty$. To handle this case, we return to (3), which now becomes

$$u' = u^2/z + 1. \tag{22}$$

Making the substitution in (6), but with

$$g(z, 1/2) = v = w = H(\theta, 1/2), \text{ where } \theta = 2\sqrt{z}, \tag{23}$$

we obtain the differential equation

$$\theta w'' + w' + \theta w = 0. \tag{24}$$

The single-valued solution of (24) that satisfies the appropriate boundary conditions is the Bessel function of order zero:

$$H(\theta, 1/2) = J_0(\theta). \tag{25}$$

Therefore, $v = g(z,\ 1/2) = J_0(2\sqrt{z})$. Since $\frac{d}{d\theta} J_0(\theta) = -J_1(\theta)$, thus,

$$\frac{d}{dz} J_0(2\sqrt{z}) = -z^{-1/2} J_1(2\sqrt{z}).$$

Hence

$$f(z,\ 1/2) = \frac{\sqrt{z}J_1(2\sqrt{z})}{J_0(2\sqrt{z})}. \tag{26}$$

The function $J_0(\theta)$ has a simple zero at $\theta_0 \doteq 2.4048255577$ (viz. 9.5 of [1]), which has the smallest modulus of any other zero. Therefore, reasoning as before, $y = z_0 = (\theta_0/2)^2 \doteq 1.4457964906$, and in this case also,

$$\lim_{n\to\infty} y^n c_n = 1. \tag{27}$$

Hence, the conjecture is certainly true for $0 < x \leqslant 1/2$, and *appears* true for $x > 1/2$ as well.

<u>Note</u>: The function $M(-k,\ B,\ Z)$, where k is a positive integer, is a polynomial in Z; this leads to *rational* solutions of (3), when c is a negative integer. This occurs when $x = x_k = 2k(k+1)/(2k+1)^2$, $k = 1, 2, \ldots$; letting u_k, v_k, $c_n^{(k)}$, and y_k denote the appropriate quantities (previously denoted by u, v, c_n, and y), we find:

$$v_k(z) = \exp(-z/(2k+1))\quad M(-k,\ 1,\ 2z/(2k+1)),$$

and

$$u_k(z) = \frac{z}{2k+1}\left\{\frac{2kM(-k+1,\ 2,\ 2z/(2k+1)) + M(-k,\ 1,\ 2z/(2k+1))}{M(-k,\ 1,\ 2z/(2k+1))}\right\}.$$

This leads to *algebraic* values for $c_n^{(k)}$, which facilitate the task of finding the appropriate value of y_k satisfying $\lim_{n\to\infty} y_k^n c_n^{(k)} = 1$.

For example, $x_1 = 4/9$ yields:

$$M(-1,\ 1,\ 2z/3) = 1 - 2z/3,\ v_1(z) = e^{-z/3}(1 - 2z/3),$$

$$u_1(z) = z(1 - 2z/9)(1 - 2z/3)^{-1} = \frac{z}{3} + \frac{2z/3}{1 - 2z/3} = \frac{z}{3} + \sum_{n=1}^{\infty}(2z/3)^n$$

$$= z + \sum_{n=2}^{\infty}(2z/3)^n.$$

Thus, $c_n^{(1)} = (2/3)^n$, $n \geqslant 2$, which implies $y_1 = 1.5$. Also,

$$v_2(z) = e^{-z/5}M(-2,\ 1,\ 2z/5) = e^{-z/5}(1 - 4z/5 + 2z^2/25),$$

and

$$u_2(z) = \frac{z(1 - 8z/25 + 2z^2/125)}{1 - 4z/5 + 2z^2/25} = z + \sum_{n=2}^{\infty}(2z/25)^n(p^n + q^n),$$

where $p = 5(1 + 2^{-1/2})$, $q = 5(1 - 2^{-1/2})$. Hence, $c_n^{(2)} = (2/25)^n(p^n + q^n)$, $n \geqslant 2$. Since $0 < q < p$, thus $y_2 = 25/2p = q \doteq 1.4644661$. Note that $\lim_{k\to\infty} x_k = 1/2$, so

$$\lim_{k\to\infty} y_k \doteq 1.4457964906,$$

the value obtained previously in connection with J_0.

References

1. M. Abramowitz & I. A. Stegun, Eds. *Handbook of Mathematical Functions*. 9th printing. Washington, D.C.: National Bureau of Standards AMS 55, 1970.
2. G. Pólya & G. Szegö. *Problems and Theorems in Analysis*, Vol. I. New York, Heidelberg, Berlin: Springer-Verlag, 1972.

Sum Product!

H-384 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany*
(Vol. 23, no. 1, February 1985)

Show that for $n = 0, 1, 2, \ldots$,

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} \prod_{j=0}^{k-1} \left[\left(n + \frac{1}{2}\right)^2 - j^2\right] = \frac{\sqrt{5}}{2} F_{2n+1}$$

Solution by the proposer

Let $F(a, b; c; z)$ denote the hypergeometric function defined by

$$F(a, b; c; z) = 1 + \sum_{k=1}^{\infty} \frac{c_k}{k!} z^k,$$

where

$$c_k = \frac{a(a+1) \ldots (a+k-1) b(b+1) \ldots (b+k-1)}{c(c+1) \ldots (c+k-1)}$$

We take $a = -n$, $b = n + 1$, $c = 1/2$, and $z = -1/4$. Then

$$c_k = (-1)^k 4^k \cdot \frac{k!(n+k)!}{(2k)!(n-k)!},$$

so that

$$F(-n, n+1; 1/2; -1/4) = 1 + \sum_{k=1}^{n} \frac{1}{(2k)!} \frac{(n+k)!}{(n-k)!}$$

$$= \sum_{k=0}^{n} \binom{n+k}{n-k} = \sum_{r=0}^{n} \binom{2n-r}{r} = F_{2n+1}. \tag{1}$$

The hypergeometric function satisfies the following identity [see F. G. Tricomi, *Vorlesungen uber Orthogonalreihen*, Springer Verlag, p. 151, formula (2.7)]:

$$F\left(-n, n+1; \frac{1}{2}; -\frac{1}{4}\right) = \frac{2}{\sqrt{5}} F\left(n + \frac{1}{2}, -n - \frac{1}{2}; \frac{1}{2}; -\frac{1}{4}\right). \tag{2}$$

Again, by using the above definition, we obtain

$$F\left(n + \frac{1}{2}, -n - \frac{1}{2}; \frac{1}{2}; -\frac{1}{4}\right) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{d_k}{k!} \cdot \frac{1}{4^k}$$

where

$$d_k = \left[\prod_{j=0}^{k-1} \left(\frac{1}{2} + j\right)\right]^{-1} \left[\prod_{j=0}^{k-1} \left(n + \frac{1}{2} + j\right)\right] \left[\prod_{j=0}^{k-1} \left(-n - \frac{1}{2} + j\right)\right]$$

$$= \frac{4^k k!}{(2k)!} (-1)^k \cdot \prod_{j=0}^{k-1} \left[\left(n + \frac{1}{2}\right)^2 - j^2\right].$$

Now the statement easily follows from (1) and (2). Q.E.D.

Also solved by P. Bruckman.

Gotta Have a System

H-385 *Proposed by M. Wachtel, Zurich, Switzerland*
(Vol. 23, no. 2, May 1985)

Solve the following system of equations:

I. $U^2_{f(n)} + V^2_{g(n)} - 3 \cdot U_{f(n)}V_{g(n)} = 1;$

II. $3 \cdot U_{h(n)}V_{i(n)} - (U^2_{h(n)} + V^2_{i(n)}) = 1.$

Solution by the proposer

I. 1) Let $U_{f(n)} = a$, $V_{g(n)} = b$. Then, we have $a^2 + b^2 - 3ab = 1$. It follows:
$$b = \frac{3a \pm \sqrt{5a^2 + 4}}{2}.$$

2) Now let $a = F_{2n}$, which leads to $\dfrac{3F_{2n} \pm \sqrt{5F^2_{2n} + 4}}{2}$.

3) Using the identity, Hoggatt I_{12}, $5F^2_{2n} + 4 = L^2_{2n}$, we obtain:
$$b_{1,2} = \frac{3F_{2n} \pm L_{2n}}{2} = F_{2n \pm 2}.$$

4) Hence, $F^2_{2n} + F^2_{2n \pm 2} - 3F_{2n}F_{2n \pm 2} = 1$, which is one of the solutions of the more generalized identity
$$F^2_{2n} + F^2_{2n \pm 2m} - L_{2m}F_{2n}F_{2n \pm 2m} = F^2_{2m},\ m = 1, 2, 3, \ldots, \text{ if } m = 1.$$

II. 1) Let $U_{h(n)} = a$, $V_{i(n)} = b$. Then, we have $3ab - (a^2 + b^2) = 1$. Thus:
$$b = \frac{3a \pm \sqrt{5a^2 - 4}}{2}.$$

2) Now let $a = F_{1+2n}$, which leads to $\dfrac{3F_{1+2n} \pm \sqrt{5F^2_{1+2n} - 4}}{2}$.

3) Using the identity, Hoggatt I_{12}, $5F^2_{1+2n} - 4 = L^2_{1+2n}$, we obtain:
$$b_{1,2} = \frac{3F_{1+2n} \pm L_{1+2n}}{2} = F_{1+2n \pm 2}.$$

4) Hence, $3F_{1+2n}F_{1+2n \pm 2} - (F^2_{1+2n} + F^2_{1+2n \pm 2}) = 1$,

which is one of the solutions of the more generalized identity
$$L_{2m}F_{1+2n}F_{1+2n \pm 2m} - (F^2_{1+2n} + F^2_{1+2n \pm 2m}) = F^2_{2m},\ m = 1, 2, 3, \ldots,$$
if $m = 1$.

Also solved by P. Bruckman.

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