CYCLIC COUNTING TRIOS

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In this paper, we extend the concept of mutually counting sequences discussed in [1] to the case of three sequences of the same length. Specifically, given the positive integer $n > 1$, we define three sequences,

A: $a(0), a(1), \ldots, a(n - 1)$,

B: $b(0), b(1), \ldots, b(n - 1)$,

C: $c(0), c(1), \ldots, c(n - 1)$,

where $a(i)$ is the multiplicity of i in B, $b(j)$ is the multiplicity of j in C, and $c(k)$ is the multiplicity of k in A. We call the ordered triple (A, B, C) a *cyclic counting trio*, and we make some preliminary observations:

(i) the entries in sequences A, B, and C are nonnegative integers less than n.

(ii) if $S(A) = \sum_{i=0}^{n-1} a(i)$, $S(B) = \sum_{j=0}^{n-1} b(j)$, and $S(C) = \sum_{k=0}^{n-1} c(k)$, then

$$S(A) = S(B) = S(C) = n.$$

(iii) if (A, B, C) is a cyclic counting trio, then so are (B, C, A) and (C, A, B). Such permuted trios will not be considered to be different.

We say that the cyclic counting trio (A, B, C) is *redundant* if A, B, and C are identical. In what follows, we show that there is a unique redundant trio for each $n \geqslant 7$:

$$a(0) = n - 4,\ a(1) = 2,\ a(2) = 1,\ a(n - 4) = 1,\ a(i) = 0 \text{ for all remaining } i.$$

There are also two redundant trios when $n = 4$, one when $n = 5$, and no others. Furthermore, we show that a nonredundant trio results *only* when $n = 7$:

$$a(0) = 4,\ a(1) = 1,\ a(3) = 2,\ a(2) = a(4) = a(5) = a(6) = 0;$$

$$b(0) = 3,\ b(1) = 3,\ b(4) = 1,\ b(2) = b(3) = b(5) = b(6) = 0;$$

$$c(0) = 4,\ c(1) = c(2) = c(4) = 1,\ c(3) = c(5) = c(6) = 0.$$

As a way to become familiar with the problem, we invite the interested reader to investigate the existence of cyclic counting trios when $n < 7$. We will therefore proceed under the assumption that (A, B, C) is a cyclic counting trio and that $n \geqslant 7$. For future reference, we let

$$n^* = n - \left[\frac{n}{2}\right],$$

and note that

$$n^* = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Since $n \geqslant 7$, it follows that $n^* \geqslant 4$.

I. For each $N \geqslant n^*$, $a(N) = 0$ or 1, $b(N) = 0$ or 1, and $c(N) = 0$ or 1.

If $a(N) \geqslant 2$, then N appears at least twice in B. So

$$n = S(B) \geqslant 2N \geqslant 2n^* = \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd,} \end{cases}$$

which is only possible when n is even. In this case,

$$N = n^* = \frac{n}{2} \quad \text{and} \quad a\left(\frac{n}{2}\right) = 2,$$

which implies that $n/2$ appears exactly twice in B. Thus, 0 must appear exactly $n - 2$ times in B. Then

$a(0) = n - 2$, $a\left(\frac{n}{2}\right) = 2$, and the $n - 2$ remaining entries of A are 0

$\Rightarrow c(0) = n - 2$, $c(2) = 1$, $c(n - 2) = 1$, and the $n - 3$ remaining entries of C are 0

$\Rightarrow b(0) = n - 3$, $b(1) = 2$, $b(n - 2) = 1$, and the $n - 3$ remaining entries of B are 0

$\Rightarrow a(0) = n - 3$, a contradiction.

Conclude that $a(N) = 0$ or 1, and use a similar argument to show that $b(N) = 0$ or 1 and $c(N) = 0$ or 1.

II. $a(j) = 1$ for at most one $j \geqslant n^*$, $b(k) = 1$ for at most one $k \geqslant n^*$, and $c(\ell) = 1$ for at most one $\ell \geqslant n^*$.

Let N and N' be distinct integers, each $\geqslant n^*$, and suppose that

$$a(N) = a(N') = 1.$$

Then

$$n = S(B) \geqslant N + N' > 2n^* = \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd,} \end{cases} \quad \text{a contradiction.}$$

Conclude that there is at most one $j \geqslant n^*$ such that $a(j) = 1$. Similarly, there is at most one $k \geqslant n^*$ such that $b(k) = 1$ and at most one $\ell \geqslant n^*$ such that $c(\ell) = 1$. Note that this result implies that 0 appears at least

$$n - n^* - 1 = \left[\frac{n}{2}\right] - 1$$

times in A, B, and C, so that

$$a(0) \geqslant \left[\frac{n}{2}\right] - 1,\ b(0) \geqslant \left[\frac{n}{2}\right] - 1,\ \text{and } c(0) \geqslant \left[\frac{n}{2}\right] - 1.$$

III. If $a(j) = 1$ for some $j \geqslant n^*$, then $b(0) = j$.

Assume that $a(j) = 1$ for some $j \geqslant n^*$. Then j appears exactly once in B, so that $b(j^*) = j$ for some integer j^*. This means that j^* appears j times in C.

$$\text{If } j^* \geqslant 2, \text{ then } n = S(C) \geqslant j^*j \geqslant 2j \geqslant 2n^* = \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd,} \end{cases}$$

which is only possible when n is even, $j^* = 2$, and $j = n/2$. Hence, 2 appears $n/2$ times in C, and since $n = S(C)$, it follows that 0 appears $n/2$ times in C as well. Thus, $b(0) = n/2$, $b(2) = n/2$, and the $n - 2$ remaining entries of B are 0. This implies that $a(0) = n - 2$, $a(n/2) = 2$, and the $n - 2$ remaining entries of A are 0, contradicting the assumption that $a(j) = 1$ for some $j \geqslant n^*$. Thus, either $j^* = 1$ or $j^* = 0$.

Assume that $j^* = 1$. Then $b(1) = j$, so that

$$n = S(B) \geqslant b(0) + b(1) \geqslant \left[\frac{n}{2}\right] - 1 + j \geqslant \left[\frac{n}{2}\right] - 1 + n^* = n - 1.$$

This tells us that $b(0) + b(1) = n$ or $b(0) + b(1) = n - 1$. If $b(0) + b(1) = n$, then

$b(0) = n - j$, $b(1) = j$, and the $n - 2$ remaining entries of B are 0

$\Rightarrow a(0) = n - 2$, $a(j) = 1$, $a(n - j) = 1$, and the $n - 3$ remaining entries of A are 0

[If $n - j$ and j were equal, then $a(j) = 2$, a contradiction.]

$\Rightarrow c(0) = n - 3$, $c(1) = 2$, $c(n - 2) = 1$, and the $n - 3$ remaining entries of C are 0

$\Rightarrow b(1) = 1.$

This means that $j = 1$, contradicting the fact that $j \geqslant n^* \geqslant 4$.

If $b(0) + b(1) = n - 1$, then

$$b(0) = n - j - 1,\ b(1) = j,$$

one of the remaining entries of B is 1, and the other $n - 3$ remaining entries of B are 0. If $n - j - 1 = j$, then $a(j) = 2$, a contradiction. If $n - j - 1 = 1$ or 0, then $b(0) = 1$ or 0, contradicting the fact that

$$b(0) \geqslant \left[\frac{n}{2}\right] - 1 \geqslant 2.$$

Hence, the integers 0, 1, j, and $n - j - 1$ are all distinct. This means that 1, j, and $n - j - 1$ each appear once in B, and the $n - 3$ remaining entries of B are 0. So

$a(0) = n - 3$, $a(1) = 1$, $a(n - j - 1) = 1$, $a(j) = 1$, and the $n - 4$ remaining entries of A are 0

$\Rightarrow c(0) = n - 4,\ c(1) = 3,\ c(n - 3) = 1,$ and the $n - 3$ remaining entries of C are 0

$\Rightarrow b(1) = 1.$

Once again, this means that $j = 1$, a contradiction.

Therefore, $j^* \neq 1$. Conclude that $j^* = 0$, so that if $a(j) = 1$ for some $j \geqslant n^*$, then $b(0) = j$.

IV. If $n > 7$, there exists $j \geqslant n^*$ such that $a(j) = 1$.

Assume that $a(N) = 0$ for all $N \geqslant n^*$. Since $b(0) \geqslant \left[\frac{n}{2}\right] - 1$, two possibilities exist: either $b(0) = \left[\frac{n}{2}\right] - 1$ or $b(0) = \left[\frac{n}{2}\right]$ when n is odd. $\left(\text{If } b(0) = \left[\frac{n}{2}\right] \text{ when } n \text{ is even or if } b(0) > \left[\frac{n}{2}\right], \text{ then } a(N) \neq 0 \text{ for some } N \geqslant n^*.\right)$

Suppose first that $b(0) = \left[\frac{n}{2}\right] - 1$. Then 0 appears exactly $\left[\frac{n}{2}\right] - 1$ times in C, so that there are $n - \left(\left[\frac{n}{2}\right] - 1\right) = n^* + 1$ nonzero entries in C. Consequently,

$$n = S(A) \geqslant \sum_{i=0}^{n^*} i = \frac{n^*(n^* + 1)}{2}.$$

If n is even, then this inequality becomes

$$n \geqslant \frac{\frac{n}{2}\left(\frac{n}{2} + 1\right)}{2}, \text{ which is false for even } n > 6.$$

If n is odd, then this inequality becomes

$$n \geqslant \frac{\left(\frac{n+1}{2}\right)\left(\frac{n+1}{2} + 1\right)}{2}, \text{ which is false for odd } n > 3.$$

Suppose next that $b(0) = \left[\frac{n}{2}\right]$ when n is odd. Then 0 appears exactly $\left[\frac{n}{2}\right]$ times in C, so that there are $n - \left[\frac{n}{2}\right] = n^*$ nonzero entries in C. Therefore,

$$n = S(A) \geqslant \sum_{i=0}^{n^*-1} i = \frac{(n^* - 1)n^*}{2} = \frac{\left(\frac{n+1}{2} - 1\right)\left(\frac{n+1}{2}\right)}{2},$$

which is false for odd $n > 7$.

The conclusion follows.

V. If $n = 7$, $a(N) = 0$ for all $N \geqslant n^* = 4$, and $b(0) = \left[\frac{n}{2}\right] = 3$, then two cyclic counting trios exist, one of which is nonredundant. (These represent the only set of circumstances that did *not* lead to a contradiction in IV.)

Since $b(0) = 3$ and $S(B) = 7$, it follows that

$$\sum_{k=1}^{6} b(k) = 4.$$

Furthermore, $S(C) = 7$ implies that

$$\sum_{k=1}^{6} kb(k) = 7.$$

For convenience, we will let $\{k_1, k_2, k_3, k_4, k_5, k_6\}$ represent some permutation of $\{1, 2, 3, 4, 5, 6\}$. From II, we know that

$$a(0) \geqslant \left[\frac{n}{2}\right] - 1 = 2.$$

$$a(0) = 2 \Rightarrow b(k_1) = b(k_2) = b(k_3) = b(k_4) = 1,\ b(k_5) = b(k_6) = 0$$
$$\Rightarrow 7 = k_1 + k_2 + k_3 + k_4 \geqslant 10, \quad \text{a contradiction.}$$

$$a(0) = 3 \Rightarrow b(k_1) = 2,\ b(k_2) = b(k_3) = 1,\ b(k_4) = b(k_5) = b(k_6) = 0$$
$$\Rightarrow 7 = 2k_1 + k_2 + k_3 \Rightarrow k_1 = 1,\ k_2 = 2,\ k_3 = 3$$
$$\Rightarrow b(1) = 2,\ b(2) = b(3) = 1,\ b(4) = b(5) = b(6) = 0.$$

Recalling that $b(0) = 3$, we find that

$$a(0) = 3,\ a(1) = 2,\ a(2) = a(3) = 1,\ a(4) = a(5) = a(6) = 0,$$

which, in turn, implies that

$$c(0) = 3,\ c(1) = 2,\ c(2) = c(3) = 1,\ c(4) = c(5) = c(6) = 0.$$

This is the redundant trio predicted for $n = 7$.

$$a(0) = 4 \Rightarrow b(k_1) + b(k_2) = 4,\ b(k_3) = b(k_4) = b(k_5) = b(k_6) = 0.$$

If $b(k_1) = b(k_2) = 2$, then $2k_1 + 2k_2 = 7$, a contradiction. If $b(k_1) = 3$ and $b(k_2) = 1$, then $3k_1 + k_2 = 7$, so that either $k_1 = 2$ and $k_2 = 1$ or $k_1 = 1$ and $k_2 = 4$. In the first case, $b(0) = 3$, $b(1) = 1$, $b(2) = 3$, and the four remaining entries of B are $0 \Rightarrow a(0) = 4$, $a(1) = 1$, $a(3) = 2$, and the four remaining entries of A are $0 \Rightarrow c(0) = 4$, $c(1) = 1$, $c(2) = 1$, $c(4) = 1$, and the three remaining entries of C are $0 \Rightarrow b(1) = 3$, a contradiction.

In the second case, $b(0) = 3$, $b(1) = 3$, $b(4) = 1$, and the four remaining entries of B are $0 \Rightarrow a(0) = 4$, $a(1) = 1$, $a(3) = 2$, and the four remaining entries of A are $0 \Rightarrow c(0) = 4$, $c(1) = 1$, $c(2) = 1$, $c(4) = 1$, and the three remaining entries of C are 0. This is the nonredundant trio predicted at the outset for $n = 7$.

$$a(0) = 5 \Rightarrow b(k_1) = 4,\ b(k_2) = b(k_3) = b(k_4) = b(k_5) = b(k_6) = 0$$
$$\Rightarrow 4k_1 = 7, \text{ a contradiction.}$$

$$a(0) = 6 \Rightarrow b(k_1) = b(k_2) = b(k_3) = b(k_4) = b(k_5) = b(k_6) = 0$$
$$\Rightarrow 0 = 4, \text{ a contradiction.}$$

If $n = 7$ and $a(j) = 1$ for some $j \geqslant n^* = 4$, then it is easy to verify that j must be 4. The cyclic counting trios that subsequently result are permuted versions of the nonredundant one just found. As a result, we may now continue under the assumption that $n > 7$.

VI. $a(n^* - 1) = 0$; $c(0) \geqslant \left[\frac{n}{2}\right]$.

Suppose that $a(n^* - 1) \neq 0$. Then $n^* - 1$ appears at least once in B. Since $b(0) = j$ and since $j \geqslant n^*$ implies $j \neq n^* - 1$, we find that

$$n = S(B) \geqslant j + (n^* - 1) \geqslant n^* + (n^* - 1)$$
$$= 2n^* - 1 = \begin{cases} n - 1 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

This tells us that $a(n^* - 1) = 1$, i.e., $n^* - 1$ appears exactly once in B.

If n is even, then some other entry of B is 1 and the $n - 3$ remaining entries of B are 0. Therefore,

$$a(0) = n - 3,\ a(1) = 3, \text{ and the } n - 2 \text{ remaining entries of } A \text{ are } 0$$
$$\Rightarrow c(0) = n - 2,\ c(3) = 1,\ c(n - 3) = 1, \text{ and the } n - 3 \text{ remaining entries of } C \text{ are } 0$$
$$\Rightarrow b(1) = 2, \text{ a contradiction.}$$

If n is odd, then the $n - 2$ remaining entries of B are 0. Therefore,

$$a(0) = n - 2,\ a(1) = 2, \text{ and the } n - 2 \text{ remaining entries of } A \text{ are } 0$$
$$\Rightarrow c(0) = n - 2,\ c(2) = 1,\ c(n - 2) = 1, \text{ and the } n - 3 \text{ remaining entries of } C \text{ are } 0$$
$$\Rightarrow b(1) = 2, \text{ again a contradiction.}$$

Hence, we conclude that $a(n^* - 1) = 0$. Using this fact and the observation following II, we can now assert that 0 appears at least $\left(\left[\frac{n}{2}\right] - 1\right) + 1 = \left[\frac{n}{2}\right]$ times in A, so that $c(0) \geqslant \left[\frac{n}{2}\right]$.

VII. If $c(0) = \left[\frac{n}{2}\right]$, then the only cyclic counting trio that results is the redundant one for $n = 8$.

Since $c(0) = \left[\frac{n}{2}\right]$, it follows that $a(i) \neq 0$ for $1 \leqslant i \leqslant n^* - 2$. Thus, each positive integer less than or equal to $n^* - 2$ appears at least once in B. Recalling that j appears once in B as well, we get

$$n = S(B) \geqslant j + \sum_{i=1}^{n^*-2} i \geqslant n^* + \frac{(n^* - 2)(n^* - 1)}{2},$$

i.e.,

$$n \geqslant \frac{(n^*)^2 - n^* + 2}{2}.$$

If n is odd, then $n^* = (n + 1)/2$ and this inequality leads to $n^2 - 8n + 7 \leqslant 0$, a contradiction for odd $n > 7$. If n is even, then $n^* = n/2$ and this inequality leads to $n^2 - 10n + 8 \leqslant 0$, a contradiction for even $n > 8$.

The case in which $n = 8$ produces the redundant cyclic counting trio with $a(0) = 4$, $a(1) = 2$, $a(2) = 1$, $a(4) = 1$, and $a(i) = 0$ for all remaining i.

VIII. If $c(0) > \left[\frac{n}{2}\right]$, then $b(n^* - 1) = 0$ and $a(0) \geqslant \left[\frac{n}{2}\right]$.

The fact that $c(0) > \left[\frac{n}{2}\right]$ implies that $c(0) \geqslant n^*$. Therefore, $b(k) = 1$ for exactly one integer $k \geqslant n^*$ and $c(0) = k$. If $b(n^* - 1) \neq 0$, then $n^* - 1$ appears at least once in C. Since k appears in C as well, and since

$$k + (n^* - 1) > \left[\frac{n}{2}\right] + (n^* - 1) = n - 1,$$

it follows from $S(C) = n$ that the $n - 2$ remaining entries of C must be 0 and that

$$k = c(0) = \left[\frac{n}{2}\right] + 1.$$

Thus,

$$b(0) = n - 2,\ b\left(\left[\frac{n}{2}\right] + 1\right) = 1,\ b(n^* - 1) = 1,$$
and the $n - 3$ remaining entries of B are 0

$$\Rightarrow a(0) = n - 3,\ a(1) = 2,\ a(n - 2) = 1,$$
and the $n - 3$ remaining entries of A are 0

$$\Rightarrow c(0) = n - 3,\ c(1) = 1,\ c(2) = 1,\ c(n - 3) = 1,$$
and the $n - 4$ remaining entries of C are 0, contradicting the fact that $b(0) = n - 2$.

As a result, we conclude that $b(n^* - 1) = 0$, so that (as in VI), $a(0) \geqslant \left[\frac{n}{2}\right]$.

IX. If $a(0) = \left[\frac{n}{2}\right]$, no cyclic counting trio can be produced; if $a(0) > \left[\frac{n}{2}\right]$, then $c(n^* - 1) = 0$.

The argument used in VII can be employed to show that no cyclic counting trio results when $a(0) = \left[\frac{n}{2}\right]$. (The only possibility, the redundant trio for $n = 8$, is disqualified because $c(0) > \left[\frac{n}{2}\right]$.) If $a(0) > \left[\frac{n}{2}\right]$, then $a(0) \geqslant n^*$. Thus, $c(\ell) = 1$ for exactly one integer $\ell \geqslant n^*$, and $a(0) = \ell$. As in VIII, we can conclude that $c(n^* - 1) = 0$.

At this point, we are left with one case to consider:

$$a(j) = 1,\ b(0) = j;\ b(k) = 1,\ c(0) = k;$$
$$c(\ell) = 1,\ a(0) = \ell, \text{ where } j,\ k,\ \ell \geqslant n^*.$$

X. $j = k = \ell$.

For convenience, let us write $j = n - r$, $k = n - s$, and $\ell = n - t$, where $1 \leqslant r,\ s,\ t \leqslant \left[\frac{n}{2}\right]$.

If $r = 1$, then $j = n - 1$, so $b(0) = n - 1$. This means that $n - 1$ entries of C are 0, contradicting the fact that $c(0) = k$ and $c(\ell) = 1$. If $r = 2$, then $j = n - 2$, so $b(0) = n - 2$. Since $c(0) = k$ and $c(\ell) = 1$, all remaining entries of C must be 0. Then $n = S(C) = k + 1$, implying that $k = n - 1$. Hence, $c(0) = n - 1$, so that $n - 1$ entries of A are 0, contradicting the fact that $a(0) = \ell$ and $a(j) = 1$. Therefore, $r \neq 1$ or 2. Similarly, $s \neq 1$ or 2 and $t \neq 1$ or 2.

Suppose that $a(i) \neq 0$ for some integer $i \geqslant r - 1$, where $i \neq j$. (Note that $i \geqslant 2$.) Then

$$n = S(B) \geqslant i + j + 1 \geqslant r - 1 + j + 1 = r + j = n,$$

which implies that $i = r - 1$ and that the $n - 3$ remaining entries of B are 0. Hence,

$$a(0) = n - 3,\ a(1) = 1,\ a(j) = 1,\ a(r - 1) = 1, \text{ and the } n - 4 \text{ remaining entries of } A \text{ are } 0$$
$$\Rightarrow c(0) = n - 4,\ c(1) = 3,\ c(n - 3) = 1, \text{ and the } n - 3 \text{ remaining entries of } C \text{ are } 0$$
$$\Rightarrow b(0) = n - 3,\ b(1) = 1,\ b(3) = 1,\ b(n - 4) = 1, \text{ and the } n - 4 \text{ remaining entries of } B \text{ are } 0$$
$$\Rightarrow a(0) = n - 4, \text{ a contradiction.}$$

Consequently, $a(i) = 0$ for all integers $i \geqslant r - 1$, where $i \neq j$. In a similar manner, we can show that

$$b(i) = 0 \text{ for all integers } i \geqslant s - 1, \text{ where } i \neq k,$$

and

$$c(i) = 0 \text{ for all integers } i \geqslant t - 1, \text{ where } i \neq \ell.$$

Thus,

$$c(0) \geqslant ((n-1)-(r-1)+1)-1 = n-r, \quad \Rightarrow k \geqslant j$$
$$a(0) \geqslant ((n-1)-(s-1)+1)-1 = n-s, \quad \Rightarrow \ell \geqslant k$$
$$b(0) \geqslant ((n-1)-(t-1)+1)-1 = n-t, \quad \Rightarrow j \geqslant \ell$$

These three inequalities together imply that $j = k = \ell$.

XI. A unique redundant cyclic counting trio exists for $n > 7$.

From X, we now know that for some $j \geqslant n^*$,

$$a(j) = b(j) = c(j) = 1 \quad \text{and} \quad a(0) = b(0) = c(0) = j.$$

Since $b(i) = 0$ whenever $i \geqslant r-1$ and $i \neq j$, this accounts for $n - r = j$ zeros in B. Because $a(0) = j$, it follows that $b(i) \neq 0$ for $1 \leqslant i \leqslant r-2$. Then

$$n = S(B) = j + 1 + \sum_{i=1}^{r-2} b(i),$$

which implies that

$$\sum_{i=1}^{r-2} b(i) = n - j - 1 = r - 1.$$

If $r = 3$, then $b(1) = 2$, so that B consists of one entry of $j = n - 3$, one entry of 1, one entry of 2, and $n - 3$ entries of 0. Therefore,

$$a(0) = n-3,\ a(1) = 1,\ a(2) = 1,\ a(n-3) = 1,$$
and the $n - 4$ remaining entries of A are 0

$$\Rightarrow c(0) = n - 4, \text{ contradicting the fact that } c(0) = j = n - 3.$$

So $r > 3$. Then

$$\sum_{i=1}^{r-2} b(i) = r - 1$$

implies that one of the terms in the sum is 2 and each of the $r - 3$ others is 1. Thus, B consists of one entry of j, one entry of 2, $r - 2$ entries of 1, and j entries of 0. Then

$$a(0) = j,\ a(1) = r-2,\ a(2) = 1,\ a(j) = 1,$$
and the $n - 4$ remaining entries of A are 0,

which implies that $c(0) = n - 4$.

If $j \neq n - 4$, then the resulting contradiction indicates that no cyclic counting trio can be produced; if $j = n - 4$ (i.e., if $r = 4$), we have

$$a(0) = n-4,\ a(1) = 2,\ a(2) = 1,\ a(n-4) = 1,$$
and the $n - 4$ remaining entries of A are 0

$$\Rightarrow c(0) = n-4,\ c(1) = 2,\ c(2) = 1,\ c(n-4) = 1,$$
and the $n - 4$ remaining entries of C are 0

$b(0) = n - 4$, $b(1) = 2$, $b(2) = 1$, $b(n - 4) = 1$,
and the $n - 4$ remaining entries of B are 0.

This is the previously mentioned cyclic counting trio for $n > 7$.

REFERENCE

1. S. Kahan. "Mutually Counting Sequences." *The Fibonacci Quarterly* **18**, no. 1 (1980):47-50.

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