REPRESENTING $\binom{2n}{n}$ AS A SUM OF SQUARES

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INTRODUCTION

A well-known theorem of Lagrange [4, p. 302] states that every natural number can be represented as a sum of at most four squares. For each integer, k, such that $1 \leqslant k \leqslant 4$, let S_k be the set of natural numbers, n, such that $\binom{2n}{n}$ is a sum of k (but not fewer) squares. We show that S_1 is empty, $S_2 = \{1, 3\}$, while S_3 and S_4 are both infinite.

PRELIMINARIES

Let p denote a prime.

Definition 1: $o_p(n) = k$ if $p^k \mid n$, $p^{k+1} \nmid n$

Definition 2: $t_p(n) = \sum_{i=0}^{r} a_i$ if $n = \sum_{i=0}^{r} a_i p^i$, with $0 \leqslant a_i < p$ for each i.

$$o_p(ab) = o_p(a) + o_p(b) \tag{1}$$

$$o_p(n!) = \frac{n - t_p(n)}{p - 1} \tag{2}$$

$$o_p\left(\binom{n}{k}\right) = \frac{t_p(k) + k_p(n - k) - t_p(n)}{p - 1} \tag{3}$$

$$t_p(ap^j) = t_p(a) \text{ for all } a, j \tag{4}$$

$$o_2\left(\binom{2n}{n}\right) = t_2(n) \tag{5}$$

$$n \neq a^2 + b^2 + c^2 \text{ iff } n = 2^{2k}(8m + 7) \text{ with } k \geqslant 0, m \geqslant 0 \tag{6}$$

$$n \neq a^2 + b^2 \text{ iff there is a prime, } p, \text{ such that } p \equiv 3 \pmod 4 \text{ and } o_p(n) \text{ is odd.} \tag{7}$$

Remarks: (1) follows from Definition 1. (2) is [2, p. 131, Problem 7]. (3) follows from (1) and (2). (4) follows from Definition 2. (5) follows from (3) and (4). (6) is stated in [4, p. 311]. (7) is [4, p. 299, Theorem 366]. $t_2(n)$ is denoted $\#_1(n)$ in [5].

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THE MAIN THEOREMS

Theorem 1: If $n \neq 1, 3$, then there is a prime, p, such that $p \equiv 3 \pmod 4$ and $n < p < 2n$.

Proof: Breusch [1] proved the conclusion for $n \geqslant 7$. If $n = 2$, then $p = 3$; if $4 \leqslant n \leqslant 6$, then $p = 7$.

Theorem 2: S_1 is empty; $S_2 = \{1, 3\}$.

Proof: If $2 \leqslant n < p < 2n$, then $2n < 2p$, so $o_p\left(\binom{2n}{n}\right) = 1$. Therefore, (7) and Theorem 1 imply $S_1 \cup S_2 \subseteq \{1, 3\}$. Since

$$\binom{2}{1} = 1^2 + 1^2, \text{ and } \binom{6}{3} = 4^2 + 2^2,$$

the conclusion now follows.

Remark: That S_1 is empty also follows from the theorem of P. Erdos [3], which states that $\binom{n}{k}$ is not a power if $k > 3$.

Definition 3: If $n = 2^k m$, $k \geqslant 0$, m odd, then $f(n)$ is the least positive residue of $m \pmod 8$.

Lemma 1: If m is odd, then $f(m) \equiv m \pmod 8$.

Proof: The proof follows from the hypothesis and Definition 3.

Lemma 2: If $f(a) \equiv f(b) \pmod 8$, then $f(a) = f(b)$.

Proof: The proof follows from the hypothesis and Definition 3.

Lemma 3: $f(ab) \equiv f(a)f(b) \pmod 8$.

Proof: Let $a = 2^c j$, $b = 2^d k$, with $c \geqslant 0$, $d \geqslant 0$, jk odd. Lemma 1 implies

$$f(jk) \equiv jk \equiv f(j)f(k) \pmod 8.$$

Now $f(ab) = f(2^{c+d}jk) = f(jk)$, while $f(a)f(b) = f(j)f(k)$, so

$$f(ab) \equiv f(a)f(b) \pmod 8.$$

Lemma 4: If $f(b) = 1$, then $f(ab) = f(a)$.

Proof: The proof follows from the hypothesis and Lemmas 3 and 2.

Lemma 5: $f(n^2) = 1$.

Proof: If $n = 2^k m$, $k \geqslant 0$, m odd, then $f(n^2) = f(2^{2k}m^2) = f(m^2)$. Now, Lemma 1 implies $f(m^2) \equiv m^2 \equiv 1 \pmod 8$. But $f(1) = 1$, so we have $f(n^2) \equiv f(1) \pmod 8$. Now, Lemma 2 implies $f(n^2) = f(1) = 1$.

Lemma 6: $f\left(\binom{2n}{n}\right) = f((2n)!)$.

Proof: The proof follows from Lemmas 4 and 5, since $(2n)! = \binom{2n}{n}(n!)^2$.

Definition 4: Let $g(n) = f(n!)$.

Table 1 lists $g(n)$ and $t_2(n)$ for each n such that $1 \leqslant n \leqslant 200$.

Table 1

n	$g(n)$	$t_2(n)$	n	$g(n)$	$t_2(n)$	n	$g(n)$	$t_2(n)$	n	$g(n)$	$t_2(n)$
1	1	1	51	5	4	101	1	4	151	7	5
2	1	1	52	1	3	102	3	4	152	5	3
3	3	2	53	5	4	103	5	5	153	5	4
4	3	1	54	7	4	104	1	3	154	1	4
5	7	2	55	1	5	105	1	4	155	3	5
6	5	2	56	7	3	106	5	4	156	5	4
7	3	3	57	7	4	107	7	5	157	1	5
8	3	1	58	3	4	108	5	4	158	7	5
9	3	2	59	1	5	109	1	5	159	1	6
10	7	2	60	7	4	110	7	5	160	5	2
11	5	3	61	3	5	111	1	6	161	5	3
12	7	2	62	5	5	112	7	3	162	5	3
13	3	3	63	3	6	113	7	4	163	7	4
14	5	3	64	3	1	114	7	4	164	7	3
15	3	4	65	3	2	115	5	5	165	3	4
16	3	1	66	3	2	116	1	4	166	1	4
17	3	2	67	1	3	117	5	5	167	7	5
81	3	2	68	1	2	118	7	5	168	3	3
91	1	3	69	5	3	119	1	7	169	3	4
20	5	2	70	7	3	120	7	4	170	7	4
21	1	3	71	1	4	121	7	5	171	5	5
22	3	3	72	1	2	122	3	5	172	7	4
23	5	4	73	1	3	123	1	6	173	3	5
24	7	2	74	5	3	124	7	5	174	5	5
25	7	3	75	7	4	125	3	6	175	3	6
26	3	3	76	5	3	126	5	6	176	1	3
27	1	4	77	1	4	127	3	7	177	1	4
28	7	3	78	7	4	128	3	1	178	1	4
29	3	4	79	1	5	129	3	2	179	3	5
30	5	4	80	5	2	130	3	2	180	7	4
31	3	5	81	5	3	131	1	3	181	3	5
32	3	1	82	5	3	132	1	2	182	1	5
33	3	2	83	7	4	133	5	3	183	7	6
34	3	2	84	3	3	134	7	3	184	1	4
35	1	3	85	7	4	135	1	4	185	1	5
36	1	2	86	5	4	136	1	2	186	5	5
37	5	3	87	3	5	137	1	3	187	7	6
38	7	3	88	1	3	138	5	3	188	1	5
39	1	4	89	1	4	139	7	4	189	5	6
40	5	2	90	5	4	140	5	3	190	3	6
41	5	3	91	7	5	141	1	4	191	5	7
42	1	3	92	1	4	142	7	4	192	7	2
43	3	4	93	5	5	143	1	5	193	7	3
44	1	3	94	3	5	144	1	2	194	7	3
45	5	4	95	5	6	145	1	3	195	5	4
46	3	4	96	7	2	146	1	3	196	5	3
47	5	5	97	7	3	147	3	4	197	1	4
48	7	2	98	7	3	148	7	3	198	3	4
49	7	3	99	5	4	149	3	4	199	5	5
50	7	3	100	5	3	150	1	4	200	5	3

Theorem 3: $\binom{2n}{n} \neq a^2 + b^2 + c^2$ iff $t_2(n)$ is even and $g(2n) = 7$.

Proof: The proof follows from (5), (6), Lemma 6, and Definition 4.

Theorem 4: Let k be a nonnegative integer. Then

(a) $g(8k) = g(4k)$; (b) $g(8k + 2) = g(4k + 1)$;

(c) $g(8k + 4) \equiv 3g(4k + 2) \pmod 8$; (d) $g(8k + 6) = 8 - g(4k + 3)$.

Proof of (a): By Definition 4 and Lemma 4, it suffices to show that

$$f\left(\frac{(8k)!)}{(4k)!)}\right) = 1 \text{ for all } k \geqslant 0.$$

We proceed by induction on k. The statement is trivially true for $k = 0$. Now

$$f\left(\frac{(8(k+1)!)}{(4(k+1)!)}\right) = f\left(\frac{(8k+8)!)}{(4k+4)!)}\right) = f\left(\frac{(8k+8)!(4k)!(8k)!}{(8k)!(4k+4)!(4k)!}\right)$$

$$= f\left(\frac{(8k+8)!(4k)!}{(8k)!(4k+4)!}\right)$$

by induction hypothesis and Lemma 4. But

$$f\left(\frac{(8k+8)!(4k)!}{(8k)!(4k+4)!}\right)$$

$$= f\left(\frac{(8k+8)(8k+7)(8k+6)(8k+5)(8k+4)(8k+3)(8k+2)(8k+1)}{(4k+4)(4k+3)(4k+2)(4k+1)}\right)$$

$$= f(2^4(8k+7)(8k+5)(8k+3)(8k+1) = f(7 \cdot 5 \cdot 3 \cdot 1) = f(105) = 1.$$

Parts (b), (c), and (d) may be proved in similar fashion.

Theorem 5: $g(2m) = \begin{cases} g(m) & \text{if } m \equiv 1 \pmod 4, \\ 8 - g(m) & \text{if } m \equiv 3 \pmod 4. \end{cases}$

Proof: The proof follows from Theorem 4.

Theorem 6: If either (i) $m \equiv 1 \pmod 4$ and $g(m) = 5$, or (ii) $m \equiv -1 \pmod 4$ and $g(m) = 3$, then $g(2m) = 5$ and $g(4m) = 7$.

Proof: The hypothesis and Theorem 5 imply $g(2m) = 5$. Now $m = 4r \pm 1$, so

$$g(4m) = g(4(4r \pm 1)) = g(8(2r) \pm 4) \equiv 3g(4(2r) \pm 2) \equiv 3g(2(4r \pm 1)),$$

$$3g(2m) \equiv 3 \cdot 5 \equiv 7 \pmod 8,$$

by Theorem 4(c). Therefore, $g(4m) = 7$.

Theorem 7: If m is odd and $g(2m) = 5$, then $g(2^k m) = 7$ for all $k \geqslant 2$.

Proof: (Induction on k.) By Theorem 6, the statement is true for $k = 2$. If $k > 2$, then $g(2^k m) = g(8(2^{k-3}m)) = g(4(2^{k-3}m)) = g(2^{k-1}m) = 7$, by Theorem 4(a) and the induction hypothesis.

Theorem 8: S_3 is infinite, that is, there exist infinitely many n such that

$$\binom{2n}{n} = a^2 + b^2 + c^2.$$

Proof: If $m \geqslant 2$, then $t_2(2^{2m-1} - 1) = 2m - 1$, and $2^{2m-1} - 1 > 3$, so that Theorems 2 and 3 imply that $2^{2m-1} - 1$ belongs to S_3.

Theorem 9: S_4 is infinite, that is, there exist infinitely many n such that

$$\binom{2n}{n} \neq a^2 + b^2 + c^2.$$

Proof: By Theorems 3, 6, and 7, it suffices to find an m such that (i) $t_2(m)$ is even, and either (ii) $m \equiv 1 \pmod 4$ and $g(m) = 5$, or (iii) $m \equiv 3 \pmod 4$ and $g(m) = 3$. Examining Table 1, we find the following such $m < 200$:

$$m \in \{3, 15, 43, 53, 63, 147, 153, 175, 189\}.$$

Concluding Remarks: Let d_n be the asymptotic density of S_n, where $1 \leqslant n \leqslant 4$. Since $S_1 \cup S_2$ is finite, by Theorem 2, we have $d_1 = d_2 = 0$, so that $d_3 + d_4 = 1$. If n is a randomly chosen natural number, let A be the event that $t_2(n)$ is even; let B be the event that $g(2n) = 7$. It is easily seen that $\Pr(A) = \frac{1}{2}$. Now $d_4 = \Pr(n \in S_4) = \Pr(A \cap B) \leqslant \Pr(A) = \frac{1}{2}$. Therefore, $d_3 \geqslant \frac{1}{2}$. Table 1 suggests that A and B are independent, and that $\Pr(B) = \frac{1}{4}$. Therefore,

Conjecture: $d_4 = 1/8$, $d_3 = 7/8$.

REFERENCES

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